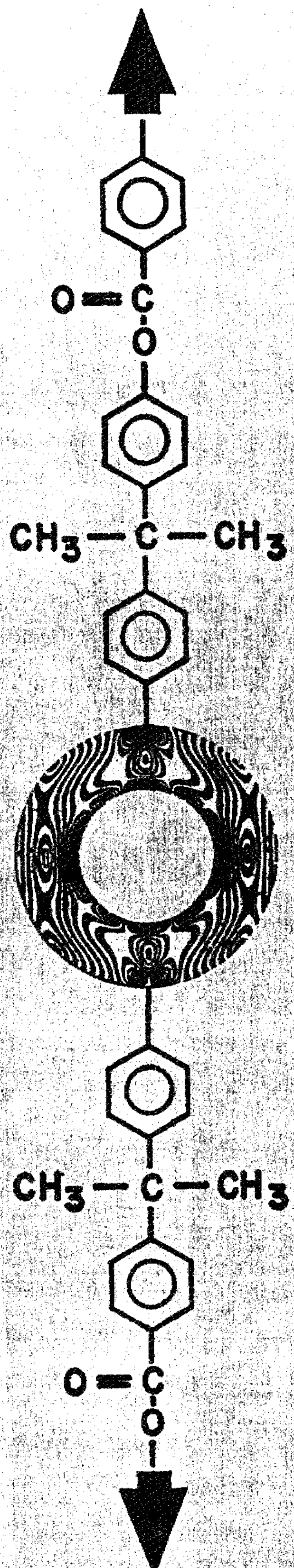

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FUNDAMENTAL CONSEQUENCES OF A NEW INTRINSIC TIME MEASURE. PLASTICITY AS A LIMIT OF THE ENDOCHRONIC THEORY

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Abstract

A new measure of intrinsic time is introduced which broadens the endochronic theory and lends it a wider predictive scope. Idealized plastic models are shown to be constitutive subsets of the general theory and the phenomenon of yield is proved to be a consequence of a particular definition of the intrinsic time measure in terms of the plastic strain tensor.

Various versions of the classical plasticity theory are shown to be asymptotic cases of the endochronic theory. In particular the kinematic hardening model, the isotropic hardening model as well as their combinations, are derivable directly from the general theory. In addition, the translation vector of the yield surface in stress space is found to be a constitutive property given by a linear functional of the history of the plastic strain. The Prager and Ziegler rules are immediately obtainable as special cases.

It is also shown that the essential features of plastic response under conditions of "stress reversals" are contained in the constitutive equation whose form remains invariant of the deformation history. It is believed that this is the first time that one single constitutive equation has been shown to predict correctly the essential features of plastic response under conditions of loading, unloading and reloading. However, when the intrinsic time measure is precisely equal to the norm of the increment of the plastic strain tensor, the constitutive unity, spoken of above, no longer exists. In this event loading and unloading cases must be distinguished.

1. Introduction

The concept of the intrinsic time scale was introduced as a proper base of measurement of the memory of a material of its past deformation history, leading to constitutive theories which we have called "endochronic".[1,2] The case of strain rate independent yet history dependent materials was dealt with at length in previous references by the author and subsequently by other workers in the field whose contributions are duly referenced.[3-8] Other work, in other directions, involving the inelastic behavior of metals is currently being pursued by other authors. References 9, 10 and 11 are typical of this work. In the papers cited heretofore, we introduced two measures of intrinsic time. One pertains to the path traced by the deformation state in a nine dimensional strain space. The other pertains to a stress path traced by the state of stress in a nine dimensional stress space. If we denote these times by ζ_c and ζ_σ respectively, then:

$$d\zeta_c^2 = P_{ijkl}\, dE_{ij}\, dE_{k\ell} \tag{1.1}$$

$$d\zeta_\sigma^2 = R_{\alpha\beta\gamma\delta}\, d\pi^{\alpha\beta}\, d\pi^{\gamma\delta} \tag{1.2}$$

where $\underset{\sim}{E}$ and $\underset{\sim}{\pi}$ are the Green strain and Piola stress tensors respectively and $\underset{\approx}{P}$ and $\underset{\approx}{R}$ are the metrics of the corresponding spaces. Specifically,

$$\pi^{\alpha\beta} = \frac{\rho}{\rho_o} \frac{\partial \chi^\alpha}{\partial y_i} \frac{\partial \chi^\beta}{\partial y_j} T_{ij} \tag{1.3}$$

where $\underset{\sim}{T}$ is the Cauchy stress tensor. Repeated indices imply summation unless otherwise stated.

The constitutive equations that followed, in the light of these definitions, were obtained from thermodynamic arguments. In particular, the

internal variable theory was used to arrive at the following set of constitutive equations, which pertain to isothermal conditions,

$$\underset{\sim}{\pi} = \frac{\partial\psi}{\partial \underset{\sim}{E}} \tag{1.4}$$

$$\frac{\partial\psi}{\partial \underset{\sim}{q}_r} + \underset{\approx}{b}_r \cdot \frac{d\underset{\sim}{q}_r}{dz} = 0 \qquad \text{(r not summed)} \tag{1.5}$$

$$z = z(\zeta) \quad , \quad \frac{dz}{d\zeta} > 0 \tag{1.6}$$

where ψ is the free energy density and $\underset{\sim}{q}_r$ are the internal variables of the thermodynamic system. In eq. (1.5) $\underset{\approx}{b}_r$ is a dissipation tensor which is at least positive semidefinite - but without loss of generality may be viewed as positive definite since otherwise, if $\underset{\approx}{b}_r = \underset{\approx}{0}$, the corresponding r'th internal variable may be eliminated from the equations as a consequence of the resulting condition:

$$\frac{\partial\psi}{\partial \underset{\sim}{q}_r} = 0 \quad . \tag{1.7}$$

The rate of irreversible entropy production which is a measure of the internal dissipation is obtained from the equation

$$T\hat{\gamma} = -\frac{\partial\psi}{\partial \underset{\sim}{q}_r} \cdot \hat{\underset{\sim}{q}}_r \tag{1.8}$$

where a roof over a symbol indicates differentiation with respect to z and T is the absolute temperature. Evidently as a result of eqs. (1.5) and (1.8)

$$T\hat{\gamma} = \hat{\underset{\sim}{q}}_r \cdot \underset{\approx}{b}_r \cdot \hat{\underset{\sim}{q}}_r \qquad (r,\ \text{summed}) \tag{1.9}$$

where the right-hand side of eq. (1.9) is an inner product, clearly a positive scalar.

Observation of accumulated results of the application of the theory to a variety of histories indicates certain broadly consistent trends. The theory is evidently simple, versatile and has powers of prediction additional to the theories of plasticity of the classical type <u>provided</u> there are no reversals in the rate of stress. In particular, in one dimension, the slope of unloading at a point as the uniaxial stress strain curve was predicted by the "linear" version of the theory to be $2E_o - E_t$ where E_o is the elastic modulus and E_t the tangent modulus. This is an over-estimate of the normally observed unloading slope which is close to E_o.

The purpose of this paper is to eliminate this deficiency of the endochronic theory by introducing a measure of intrinsic time which is more closely representative of the dissipation properties of metals. The underlying cause of this discrepancy is of a thermodynamic nature and has to do with the fact that the theory, heretofore, predicts a rate of dissipation during loading which is identical to that which occurs at the onsent of unloading.

In this paper we introduce a new concept of intrinsic time ζ which corrects both these deficiencies. In one (axial) dimension we stipulate that

$$d\zeta = \left| d\varepsilon - k\,\frac{d\sigma}{E_o} \right| \tag{1.10}$$

where k is a positive scalar such that $0 \leq k \leq 1$ and E_o is the elastic modulus. If, at the onset of unloading, we denote the unloading slope by E_-

and the rate of dissipation by $\dot{\gamma}_-$, then it is shown that E_- tends to E_o and $\dot{\gamma}_-$ tends to zero as k tends to unity.

In three dimensions we form a strain-like tensor θ_{ij} given by eq. (1.11)

$$\theta_{ij} = \varepsilon_{ij} - \phi_{ij\kappa\ell}\,\sigma_{\kappa\ell} \tag{1.11}$$

where $\underset{\approx}{\phi}$ is a positive definite symmetric fourth order material tensor. Dealing strictly with isotropic materials where $\underset{\approx}{\phi}$ is isotropic we proceed to define a deviatoric strain-like tensor $\mathcal{Q}_{ij}$ by the equation:

$$\mathcal{Q}_{ij} = e_{ij} - \frac{k_1}{2\mu}\, s_{ij} \tag{1.12}$$

where e_{ij} and s_{ij} are the deviatoric strain and stress tensors respectively. We also define a hydrostatic strain-like tensor $\theta_{\kappa\kappa}$ by the equation

$$\theta_{\kappa\kappa} = \varepsilon_{\kappa\kappa} - \frac{k_o}{3K}\,\sigma_{\kappa\kappa} \tag{1.13}$$

The relation between k_o and k_1 on one hand and the components of $\underset{\approx}{\phi}$ on the other is shown in Section 6. Evidently, $\mathcal{Q}_{ij}$ and $\theta_{\kappa\kappa}$ are the deviatoric and hydrostatic components, respectively, of θ_{ij}.

A hydrostatic intrinsic time measure $d\zeta_H$ and a deviatoric counterpart $d\zeta_D$ are now defined by the eq.'s

$$d\zeta_H^2 = \kappa_{oo}\, d\theta_{\kappa\kappa}\, d\theta_{\ell\ell} + \kappa_{o1}\, d\,\mathcal{Q}_{ij}\, d\,\mathcal{Q}_{ij} \tag{1.14a}$$

$$d\zeta_H^2 = \kappa_{1o}\, d\theta_{\kappa\kappa}\, d\theta_{\ell\ell} + \kappa_{11}\, d\,\mathcal{Q}_{ij}\, d\,\mathcal{Q}_{ij}\ . \tag{1.14b}$$

where κ_{rs} are material parameters which provide for a coupling between hydrostatic and shear response.

The resulting constitutive equation for isotropic materials is given

by eq.'s (6.8) and (6.9). These are identical in form to those of the simple endochronic theory (where both k_o and k_1 are equal to zero) but predict behavior that is far closer to that of metals, when k_o and k_1 are close to unity.

The very significant particular case: $k_1 = 1$, $z_H = 0$.

This is the case of elastic hydrostatic response and the intrinsic time measure $d\zeta$ is equal to the norm of the plastic strain tensor. The resulting constitutive equation is eq. (6.36) or, equivalently, (6.38). With the aid of these equations the following propositions are proved in Section 6 of the text.

(i) A spherical yield surface in deviatoric stress space exists.

(ii) If $\rho_1 = 0$ in eq. (6.36) and $f(\zeta)$ increases monotonically with ζ, then the classical theory of plasticity with isotropic hardening follows. The increment of plastic strain is shown to be normal to the yield surface.

(iii) If $\rho_1 \neq 0$ (in which case $\rho_1 > 0$) and $f(\zeta) = 1$, then the spherical yield surface translates in deviatoric stress space and a theory of kinematic hardening results with a general rule which contains Prager's and Ziegler's rules as special cases.

(iv) If $\rho_1 \neq 0$ and $f(\zeta)$ is a monotonically increasing function then the yield surface translates and expands simultaneously according to rules inherent in the theory.

(v) The above are true in the case of softening, instead of hardening, when $f(\zeta)$ is a monotonically decreasing function.

Remark: No yield surface exists if $\kappa_1 < 1$. Thus the classical theory

of plasticity is the "boundary" of the endochronic theory.

2. The Rate of Entropy Production

For the purposes of concentrating on the physical aspects of the problem and to rid our argument of superfluous analytical complications, we limit the discussion in this Section to a one dimensional stress field and to theories with one scalar internal variable. To this end we let

$$\psi = \tfrac{1}{2}E\,(\varepsilon - q)^2 \tag{2.1}$$

$$\sigma = \frac{\partial\psi}{\partial\varepsilon}\,E(\varepsilon - q) \tag{2.2}$$

$$\frac{\partial\psi}{\partial q} + b\hat{q} = 0 \tag{2.3}$$

or,

$$\sigma = b\hat{q} \tag{2.4}$$

where ε is the uniaxial strain and σ the appropriate axial stress. No limitation to small deformation is therefore implied except in so far as the validity of the algebraic forms of eqs. (2.1) and (2.3) are concerned. It follows from eq. (1.9) that

$$T\,\frac{d\gamma}{dz} = \frac{1}{b}\,\sigma^2 \quad . \tag{2.5}$$

A typical unloading response, given by the linear version of the endochronic theory, using eqs. (2.2) and (2.3) is shown in Figure 1. The slope at the point of unloading (point A) is $2E_o - E_t$, in variance with the observed

slope which is in the vicinity of E_o. The reason for the discrepancy may well lie in the expression for the dissipation, i.e. eq. (2.5). The inference is that the rate of dissipation at A is the same whether one proceeds to load or unload at A.

This, unfortunately, is not in accord with experimentally established mechanisms of plastic flow. For instance, in single crystals the latter occurs during loading by cumulative slip along planes between which the crystal suffers only elastic distortion.[12] During initial unloading, back slip is minimal and the crystals change shape mainly (if not totally) by elastic recovery of the parts of the crystal between slip planes. This indicates that the rate of dissipation, if not zero during unloading, should at least be much less than its counterpart during loading.

The endochronic theory cannot predict such a result unless the increment of intrinsic time corresponding to a decrement $d\varepsilon_-$ is much less than that due to an increment $d\varepsilon_+$ such that

$$|d\varepsilon_-| = |d\varepsilon_+| \tag{2.6}$$

If indeed

$$d\zeta = k|d\varepsilon| \tag{2.7}$$

where

$$k = k(\varepsilon, \sigma, \zeta) \tag{2.8}$$

a prediction of this type of dissipation behavior is impossible. This

observation lies at the root of the difficulty that one encounters in using the endochronic theory to describe unloading.

Interestingly enough, quasilinear versions of the theory have been proposed[5] which correct the slope of the unloading response. However, the thermodynamic relation

$$d\gamma_+ = d\gamma_- \tag{2.9}$$

in obvious notation, remains intact and equally inapplicable to metals.*

*To avoid this difficulty other authors[8, 14] have proposed one type of constitutive equation for loading and another for unloading. Here we show that the difficulty is circumvented without this dichotomy.

The situation is similar when many internal variables are used. In this case the expression for ψ is a generalization of eq. (2.1):

$$\psi = \frac{1}{2} \sum_r E_r (\varepsilon - q_r)^2 \tag{2.10}$$

correspondingly,

$$\sigma = \sum_r E_r (\varepsilon - q_r) \tag{2.11}$$

$$\frac{\partial\psi}{\partial q_r} + b_r \hat{q}_r = 0 \tag{2.12}$$

$$T \frac{d\gamma}{dz} = -\sum_r \frac{\partial\psi}{\partial q_r} \hat{q}_r \quad . \tag{2.13}$$

This last equation in conjunction with eq. (2.12) shows that

$$T \frac{d\gamma}{dz} = \sum_r \frac{1}{b_r} \left(\frac{\partial\psi}{\partial q_r}\right)^2 \quad . \tag{2.14}$$

The important conclusion from the right-hand side of eq. (2.14) is that $\frac{d\gamma}{dz}$ is a function of state, i.e., of the state variables ε and q_r. Thus if

$$dz_+ = dz_- \tag{2.15}$$

when $|d\varepsilon|_+ = |d\varepsilon|_-$, then

$$d\gamma_+ = d\gamma_- \tag{2.16}$$

as before.

From the above discussion it is fairly conclusive that the unloading behavior of metals will be predicted more precisely if $dz_{-} < dz_{+}$ at the point of unloading, for the same value of $|d\varepsilon|$.

3. Physical Considerations

The essential premise of the endochronic theory is that the intrinsic time increment is a measure of irreversibility. For instance, reference to eq. (2.3) shows that

$$dq = -\frac{1}{b}\frac{\partial\psi}{\partial q}\, dz \ . \tag{3.1}$$

Thus if in the course of a change in the thermodynamic state of a system dz is zero, then it follows from eq. (3.1) that dq is also zero, in which case the system is behaving reversibly.[13] Since at the point of unloading, the process of unloading borders on the reversible, though it may not be exactly so, one is led to conclude that in one dimension the ratio $\frac{dz_{-}}{|d\varepsilon|}$ must be close to zero. Furthermore, since during elastic unloading in one dimension the strain decrement is governed essentially by the relation

$$d\varepsilon = \frac{d\sigma}{E_o} \tag{3.2}$$

one is led to the conclusion that a definition of intrinsic time which is more closely representative of unloading behavior should be given by the relation

$$d\zeta = \left|d\varepsilon - k\,\frac{d\sigma}{E_o}\right| \tag{3.3}$$

where k is a scalar which in the case of metals is close to and possibly equal to unity and certainly non-negative. Thus:

$$0 \leq k \leq 1. \tag{3.4}$$

Note that except for a scalar multiplier this definition of intrinsic time reduces to our previous one,[1] when $k = 0$.

It will be shown presently that as k approaches unity the constitutive behavior as depicted by a single internal variable, approaches that of an elastic perfectly plastic solid.

4. Application of the Theory to One Dimension

We proceed to apply the theory to those materials which do not harden cyclically. In this case $z = \zeta$.

Elimination of the internal variable q from eqs. (2.2) and (2.3) leads to the following constitutive equation in one dimension:

$$\left| d\varepsilon - k\frac{d\sigma}{E_o} \right| \sigma = b\left(d\varepsilon - \frac{d\sigma}{E_o}\right) \quad . \tag{4.1}$$

We shall proceed to study materials for which $d\varepsilon > \frac{d\sigma}{E_o}$. In this case the sign of $d\varepsilon - k\frac{d\sigma}{E_o}$ is the same as the sign of $d\varepsilon$.

We shall consider the cycle where loading begins at $(\sigma, \varepsilon) = (0,0)$ and continues into the plastic range where unloading begins, followed by compression into the plastic range and unloading to zero stress.

In Figure 3, we show a plot of $\frac{d\sigma}{d\varepsilon}$ vs $\int |d\varepsilon|$ for a perfectly plastic solid for the purposes of comparison with the predictions of the constitutive equation (4.1).

In the course of the paths OYA and CDEA of the ideal solid of Figure 3, eq. (4.1) will read

$$d\varepsilon - k\,\frac{d\sigma}{E_o} = b\,\left(d\varepsilon - \frac{d\sigma}{E_o}\right) \quad . \tag{4.2}$$

It follows from this equation that

$$\frac{1}{E_o}\,\frac{d\sigma}{d\varepsilon} = \frac{1 - (\sigma/b)}{1 - k(\sigma/b)} \quad . \tag{4.3}$$

As σ increases the numerator will vanish before the denominator at $b = \sigma$ since $k < 1$.* Furthermore, it can be easily verified that as $k \rightarrow 1$, the slope $\frac{d\sigma}{d\varepsilon}$ of the stress response approaches that of a perfectly plastic solid given in Figures 2 and 3, for the paths OYA and CD.

In the course of the unloading-compression path ABC of Figure 2, eq. (4.1) will read

$$\frac{1}{E_o}\,\frac{d\sigma}{d\varepsilon} = \frac{1 + (\sigma/b)}{1 + k(\sigma/b)} \tag{4.4}$$

which predicts a behavior which will also tend to that of the perfectly plastic solid as $k \rightarrow 1$.

Also of interest is that eqs. (4.3) and (4.4) give stress responses in tension and compression which are mirror images of each other as one can readily verify by substituting minus σ for σ in eq. (4.4).

Integration of the constitutive equation (4.1) for the two conditions $d\varepsilon > 0$ and $d\varepsilon < 0$ gives the following two equations respectively for the

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* The case of $k = 1$ must be treated as a limiting case otherwise one is lead to erroneous results.

uniaxial stress response:

$d\varepsilon > 0$:

$$\frac{E_o}{b}(\varepsilon-\varepsilon_o) = k\,\frac{\sigma-\sigma_o}{b} - (1-k)\log\frac{b-\sigma}{b-\sigma_o} \tag{4.5}$$

$d\varepsilon < 0$:

$$\frac{E_o}{b}(\varepsilon-\varepsilon_o) = k\,\frac{\sigma-\sigma_o}{b} + (1-k)\log\frac{b+\sigma}{b+\sigma_o} \tag{4.6}$$

where (ε_o,σ_o) is the initial point on the path of the appropriate solution. The essential constants of eq. (4.5) are the initial modulus E_o and the ultimate stress b. In Figure 4 we give a stress response curve, typical of pure aluminum where $E_o = 9.1 \times 10^6$ lb/in^2 and $b = 16.24 \times 10^3$ lb/in^2. It is quite apparent that the material response tends to that of an elastic perfectly plastic solid as $k \to 1$.

This result is quite obviously a major development in the evolution of the endochronic theory.

5. The Question of Dissipation

The difficulty that one encounters in calculating the dissipation, using the older version of the endochronic time measure given by eq. (1.1) has also been circumvented. The following expressions are easily obtained for $d\gamma_+$ and $d\gamma_-$:

$$Td\gamma_+ = \left(\frac{\sigma^2}{b}\right)\frac{1-k}{1-k\left(\frac{\sigma}{b}\right)}\,|d\varepsilon| \tag{5.1}$$

$$T d\gamma_- = (\frac{\sigma^2}{b}) \frac{1-k}{1+k(\frac{\sigma}{b})} |d\varepsilon| \quad . \tag{5.2}$$

If one takes limits of $d\gamma$ in their proper sequence, one finds that

$$\lim_{k\to 1} (\lim_{\sigma\to b} d\gamma_+) = b|d\varepsilon| \tag{5.3}$$

$$\lim_{k\to 1} (\lim_{\sigma\to b} d\gamma_-) = 0 \quad . \tag{5.4}$$

Eqs. (5.3) and (5.4) give the precise dissipation properties of an elastic perfectly plastic solid in the plastic state. In Figure 5 $\frac{d\gamma}{d\varepsilon}$ is plotted versus stress in the case of a solid where $k = .99$ on a semilogarithmic scale. Note that as the deformation increases the divergence between $\left.\frac{d\gamma}{d\varepsilon}\right|_+$ and $\left.\frac{d\gamma}{d\varepsilon}\right|_-$ increases the limiting ratio being equal to 2×10^3.

6. Generalization to Three Dimensions and n Internal Variables

We define the strain-like tensor θ_{ij} by the following equation:

$$\theta_{ij} = \varepsilon_{ij} - \phi_{ij\kappa\ell}\, \sigma_{\kappa\ell} \tag{6.1}$$

where $\underset{\sim}{\phi}$ is a positive definite symmetric fourth-order tensor. In the case of isotropic materials, $\underset{\sim}{\phi}$ is of the form:

$$\phi_{ij\kappa\ell} = \delta_{ij}\, \delta_{\kappa\ell}\, \phi_o + \phi_1\, (\delta_{i\kappa}\delta_{j\ell} + \delta_{i\ell}\delta_{\kappa j}) \quad . \tag{6.2}$$

In this event, $\underset{\sim}{\theta}$ may be decomposed into its deviatoric and hydrostatic parts as follows:

$$\theta_{ij} = \frac{1}{3}\,\theta_{\kappa\kappa}\delta_{ij} + \mathcal{Q}_{ij} \tag{6.3}$$

where

$$\theta_{kk} = \varepsilon_{kk} - \frac{k_o}{3K_o}\,\sigma_{\kappa\kappa} \tag{6.4}$$

$$\mathcal{Q}_{ij} = e_{ij} - \frac{k_1}{2\mu_o}\,s_{ij} \tag{6.5}$$

In eq. (6.5) $\underset{\sim}{e}$ and $\underset{\sim}{s}$ are the deviatoric parts of $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\sigma}$ respectively, μ_o and K_o are the elastic shear and bulk moduli of the material respectively and the constants k_o and k_1 are related to ϕ_o and ϕ_1 by the relations

$$2\phi_1 = \frac{k_1}{2\mu_o} \quad , \quad (3\phi_o + 2\phi_1) = \frac{k_o}{3K_o}$$

Various degrees of generality are now possible. For instance, mindful of the consequence that in the case of isotropic materials, undergoing small deformation, the deviatoric and hydrostatic responses are separable,(2) we may define a hydrostatic intrinsic time measure $d\zeta_H$, where

$$d\zeta_H^2 = \kappa_{oo}\,d\theta_{kk}\,d\theta_{\ell\ell} + \kappa_{01}\,d\mathcal{Q}_{ij}\,d\mathcal{Q}_{ij} \tag{6.6}$$

and a deviatoric intrinsic time measure $d\zeta_D$ where

$$d\zeta_D^2 = \kappa_{10}\,d\theta_{kk}\,d\theta_{\ell\ell} + \kappa_{11}\,d\mathcal{Q}_{ij}\,d\mathcal{Q}_{ij} \quad . \tag{6.7}$$

The hydrostatic measure $d\zeta_H$ would then be the appropriate one to use in the hydrostatic constitutive response, while $d\zeta_D$ would be used in the deviatoric constitutive response. In this case κ_{ij} $(i,j = 1,2)$ is a matrix of nondimensional scalars.

Using the formulation of Reference 1 as a point of departure, we may then write the rate independent (plastic) response of metals, in the small deformation region as follows:

$$\underset{\sim}{s} = 2 \int_0^{z_D} \mu(z_D - z_D') \frac{\partial \underset{\sim}{e}}{\partial z_D'} dz_D' \tag{6.8}$$

and

$$\sigma_{\kappa\kappa} = 3 \int_0^{z_H} K(z_H - z_H') \frac{\partial \varepsilon_{\kappa\kappa}}{\partial z_H'} dz_H' \tag{6.9}$$

where

$$dz_D = \frac{d\zeta_D}{f_D(\zeta_D)} \tag{6.10}$$

and

$$dz_H = \frac{d\zeta_H}{f_H(\zeta_H)} \quad . \tag{6.11}$$

The functions f_D and f_H are both non-negative.

6.1 Particular Cases

We wish to examine in greater detail the case where $\kappa_{oo} = 1$ and, κ_{o1} and κ_{10} are equal to zero. We also set $d\zeta_H$ identically zero thereby dealing with a material with an elastic hydrostatic response and no coupling between deviatoric and hydrostatic behavior. Now the constitutive description of the material simplifies considerably so that

$$\sigma_{\kappa\kappa} = 3\ K\ \varepsilon_{\kappa\kappa} \tag{6.12}$$

$$\tilde{s} = 2 \int_o^z \mu(z - z') \frac{\partial \tilde{e}}{\partial z'} dz' \tag{6.13}$$

where we have temporarily dropped the suffix D in z_D since no confusion is likely to result. Also we have set

$$d\zeta^2 = dQ_{ij}\ dQ_{ij} \tag{6.14}$$

$$Q_{ij} = e_{ij} - k_1 \frac{s_{ij}}{2\mu_o} \tag{6.15}$$

and

$$dz = \frac{d\zeta}{f(\zeta)} \quad , \quad f(\zeta) > 0 \quad . \tag{6.16}$$

The constant κ_{11} in eq. (6.7) does not appear in eq. (6.14) since it is now an immaterial constant and can be appended to the function $f(\zeta)$. Such redundancies in the number of constants that describe the constitutive

properties of a metal are present in the theory and can be dealt with as they arise without the need for further comment.

6.1.2 Uncoupled Hydrostatic and Deviatoric Responses

The nature of the matrix κ_{ij} merits further discussion. For instance, if it is diagonal then there is no coupling between the deviatoric and hydrostatic response. In this case a hydrostatic strain history will have no effect on the shear behavior and vice versa. On the other hand the theory does admit such an effect if κ_{ij} is not diagonal. At this time we do not have enough evidence to comment on the symmetry, or asymmetry, of κ_{ij}.

Of particular importance are the cases where the deviatoric and hydrostatic responses are uncoupled and either k_o, k_1 or both are equal to unity. We wish to discuss, at length, the case of the shear response; the arguments will apply equally well to its hydrostatic counterpart. With reference to eq. (6.8) one obtains the following relation

$$\underset{\sim}{s} = 2\mu_o \int_0^{z_D} \rho(z_D - z_D') \frac{d\underset{\sim}{Q}}{dz_D'} dz_D' \tag{6.17}$$

where

$$\mu(z) \overset{\text{def}}{=} \mu_o G(z) \quad ; \quad G(0) = 1 \quad , \tag{6.18}$$

and $\rho(z)$ is related to $G(z)$ by the integral equation

$$\rho(z) - k_1 \int_0^z \rho(z-z') \frac{dG}{dz'} dz' = G(z) \quad . \tag{6.19}$$

The solution of this equation in the case of two internal variables is given in Appendix A.

More simply, the following algebraic relation exists between the Laplace transforms $\bar{\rho}$ and $\bar{G}$ of ρ and G, where p is the Laplace transform* variable:

$$\bar{\rho} = \frac{\bar{G}}{1-k_1 p\bar{G}} \quad . \tag{6.20}$$

This specific form of the theory has the characteristic feature that it leads to a constitutive equation which gives the deviatoric stress response to the history of the deviatoric strain-like tensor $\underset{\sim}{Q}$ in terms of the path in $\underset{\sim}{Q}$ space. We point out however that there is no specific connection with an a priori existence of a yield surface. Furthermore there is no dichotomy in the constitutive representation of the loading and unloading responses. Equations (4.5) and (4.6) are derivatives of one and the same eq. (4.1).

It is of interest to note that $d\underset{\sim}{Q}$ is akin to and is equal to the deviatoric "plastic strain" increment, in classical plasticity terms, if k_1 is unity and the "elastic strain" increment is defined as $\frac{d\underset{\sim}{s}}{2\mu}$. Moreover, we will proceed to show that if $k_1 = 1$, then a yield surface exists. The proof will apply strictly to the case where $k_1 = 1$.

To this end we invoke the result of Reference 1 according to which:

$$G(z) = \sum_{r=1}^{n} G_r e^{-\alpha_r z} \tag{6.21}$$

where G_r are positive; also, α_r are all positive with the exception of α_1 which may be zero. Furthermore, since G(0) = 1 it follows that

$$\sum_{r=1}^{n} G_r = 1 \quad , \tag{6.22}$$

Let $\bar{G}(p)$ be the Laplace transform of G(z). Then as a result of eq. (6.21) $\bar{G}(p)$ is of the form

*For explanatory notes on this treatment see Appendix C.

$$\bar{G}(p) = \frac{\bar{P}(p)}{\bar{Q}(p)} \tag{6.23}$$

where

$$\bar{P} = G_1(p + \alpha_2) \ldots (p + \alpha_n) + (p + \alpha_1) G_2 \ldots (p + \alpha_n) \ldots$$
$$+ (p + \alpha_1) \ldots (p + \alpha_{n-1}) G_n \tag{6.24}$$

and

$$\bar{Q}(p) = (p + \alpha_1)(p + \alpha_2) \ldots (p + \alpha_n) \quad . \tag{6.24}$$

Evidently, $\bar{P}$ and $\bar{Q}$ are polynomials of order (n-1) and n respectively. Furthermore, the coefficient of the leading term of $\bar{P}$ is equal to unity. It follows from eqs. (6.20) and (6.23) that

$$\bar{\rho}(p) = \frac{\bar{P}(p)}{\bar{R}(p)} \tag{6.25}$$

where

$$\bar{R}(p) = \bar{Q}(p) - p\,\bar{P}(p) \quad . \tag{6.26}$$

It may be shown by direct computation that $\bar{R}(p)$ is a polynomial of order n-1, such that the coefficient of p^{n-1} is exactly the sum $\sum_{r=1}^{n} \alpha_r G_r$. Hence the ratio $\frac{\bar{P}(p)}{\bar{R}(p)}$ is not function-like, in the sense of Mikusinski, in that it contains a delta function of strength $\frac{1}{\sum_{r=1}^{n} \alpha_r G_r}$. In fact if $-\beta_r$ are the zeros of $\bar{R}$, then $\bar{\rho}(p)$ may be written as

$$\bar{\rho}(p) = \frac{1}{\sum_{r=1}^{n} \alpha_r G_r} + \sum_{r=1}^{n-1} \frac{R_r}{\beta_r + p} \tag{6.27}$$

since $\bar{\rho}(p) - \dfrac{1}{\sum_{r=1}^{n} \alpha_r G_r}$ is the ratio of two polynomials, the numerator being of degree n-2, and the denominator $\bar{R}(p)$ of degree n-1. It is shown below that if the absolute values of the zeros of $\bar{Q}(p)$ are ordered in the sense that

$$\alpha_1 < \alpha_2 < \alpha_3 \ldots < \alpha_n$$

then β_r must always satisfy the inequalities:

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \alpha_3 \ldots < \beta_{n-1} < \alpha_n$$

and are therefore all positive. The proof is elementary. Evidently

$$\bar{R}(-\alpha_1) = -\alpha_1(\alpha_2 - \alpha_1)(\alpha_3 - \alpha_1) \ldots (\alpha_n - \alpha_1) < 0$$

$$\bar{R}(-\alpha_2) = -\alpha_2(\alpha_1 - \alpha_2)(\alpha_3 - \alpha_2) \ldots (\alpha_n - \alpha_2) > 0 \quad .$$

Similarly,

$$\bar{R}(-\alpha_3) < 0$$

and generally $\bar{R}(p)$ alternates in sign at the zeros of $\bar{Q}(p)$. It follows that $\bar{R}(p)$ must vanish at the points $p = -\beta_1$, $p = -\beta_2 \ldots p = -\beta_{n-1}$ where β_r are all positive and are bounded from above and below by α_r according to the

inequalities given above. Finally, since $\bar{R}(p)$ is a polynomial of degree n-1 it can have no other zeros. This completes the proof.

The values of the residues R_r are found from the formula

$$R_r = \frac{\bar{P}(-\beta_r)}{\bar{R}'(-\beta_r)} \tag{6.28}$$

where $\bar{R}'(p) \underset{=}{\text{def}} \frac{d\bar{R}}{dp}$. For a detailed calculation see Appendix B where it is shown that the residues R_r are all positive.

Clearly, as a result of eq. (6.27)

$$\rho(z) = \frac{\delta(z)}{\sum_{r=1}^{n} \alpha_r G_r} + \sum_{r=1}^{n-1} R_r e^{-\beta_r z} \tag{6.29}$$

which we write as

$$\rho(z) = \rho_o \delta(z) + \rho_1(z) \tag{6.30}$$

where $\rho_1(z)$ is composed of a finite sum of exponential terms.

Evidently, it follows that

$$\underset{\sim}{s} = 2\mu_o \rho_o \frac{d\underset{\sim}{Q}}{dz} + 2\mu_o \int_o^z \rho_1(z-z') \frac{\partial \underset{\sim}{Q}}{dz} dz' \quad . \tag{6.31}$$

Now, as a consequence of eq. (6.31), at z = 0:

$$\underset{\sim}{s} = 2\mu_o \rho_o \left. \frac{d\underset{\sim}{Q}}{dz} \right|_{z=0} \quad . \tag{6.32}$$

Also from eq. (6.15) (and for $k_1 = 1$), the condition $\underset{\sim}{Q} = 0$ (and, therefore, z = 0) gives the relation

$$\underset{\sim}{s}' = 2\mu_o \underset{\sim}{e} \quad . \tag{6.33}$$

Equation (6.33) merely attests to the fact that while z = 0 the deformation process is reversible and therefore the deviatoric stress response is elastic. See Sec.7. It is also of interest that at z = 0, $\frac{d\underset{\sim}{Q}}{dz}$ is indeterminate and can take any value consistent with eq. (6.33). Specifically, eqs. (6.32) and (6.33) combine to give

$$\rho_o \left.\frac{d\underset{\sim}{Q}}{dz}\right|_o = \underset{\sim}{e} \quad . \tag{6.34}$$

However, at z > 0 the derivative $\frac{d\underset{\sim}{Q}}{dz}$ exists and specifically at z = 0+ one obtains the limit of $\frac{d\underset{\sim}{Q}}{dz}$ by approaching z = 0 from the right. In fact, the point 0+ is the point of deviation from elastic response or the yield point. At this point

$$\left|\frac{d\underset{\sim}{Q}}{d\zeta}\right|^2 = 1$$

in accordance with eq. (6.14). Therefore, applying eq. (6.33)

$$|\underset{\sim}{s}|^2 = 4\mu_o^2 \rho_o^2 \left(\frac{d\zeta}{dz}\right)^2 \overset{\text{def}}{\equiv} s_Y^2 \quad . \tag{6.35}$$

In conclusion, in the process of monotonic loading, while $|\underset{\sim}{s}| \leq s_Y$, z = 0 and eq. (6.33) applies and the material response is elastic. However, when $|\underset{\sim}{s}| > s_Y$, then eq. (6.31) applies and the response is no longer elastic. Of course, s_Y is the YIELD stress and eq. (6.35) is the VON MISES yield criterion.

In particular, as a result of eq. (6.35), the constitutive relation (6.31) may be written more succinctly as

$$\underset{\sim}{s} = s_Y^o \frac{d\underset{\sim}{\varrho}}{dz_D} + 2\mu_o \int_o^{z_D} \rho_1(z_D - z'_D) \frac{\partial \underset{\sim}{\varrho}}{\partial z'_D} dz'_D \tag{6.36}$$

where $s_Y^o = 2\mu_o \rho_o$ and has the physical significance of an initial yield stress.

We shall show presently that eq. (6.36) yields very rich results and reveals important characteristics of plastic behavior that first appeared as assumptions or conjectures in the classical theory of plasticity.

To this end, let the integral on the right-hand side of eq. (6.36) be denoted by $\underset{\sim}{r}$, i.e., set:

$$\underset{\sim}{r} = 2\mu_o \int_o^{z_D} \rho_1(z_D - z'_D) \frac{\partial \underset{\sim}{\varrho}}{\partial z'_D} dz'_D \tag{6.37}$$

Equation (6.36) then reads,

$$\underset{\sim}{s} - \underset{\sim}{r} = s_Y^o \frac{d\underset{\sim}{\varrho}}{d\zeta_D} f(\zeta_D) \tag{6.38}$$

We recall that $k_1 = 1$ and therefore $d\underset{\sim}{\varrho}$ is exactly equal to the increment of plastic strain. We wish to examine eq. (6.38) in the specific case where ζ_D and ζ_H are uncoupled,

$$d\zeta_D^2 = d\underset{\sim}{\varrho} \cdot d\underset{\sim}{\varrho} \tag{6.39}$$

and the hydrostatic response is elastic, i.e., $dz_H = 0$, in which event eq. (6.9) gives rise to eq. (6.12), i.e.

$$\sigma_{KK} = 3K\varepsilon_{KK} \tag{6.40}$$

in our previous notation.

Case (i): $f(\zeta) = 1$

Equations (6.38) and (6.39) give rise immediately to the following two results:

$$||\underset{\sim}{s} - \underset{\sim}{r}||^2 = s_Y^{o\,2} \tag{6.41}$$

$$d\underset{\sim}{\ell} = \frac{1}{s_Y^o}(\underset{\sim}{s} - \underset{\sim}{r})\, d\zeta_D \tag{6.42}$$

It is clearly obvious that if $\underset{\sim}{r} = \underset{\sim}{0}$ then eq. (6.42) is that of an elastic perfectly plastic material with a Von Mises yield criterion. In this case

$$d\underset{\sim}{\ell} = \frac{1}{s_Y^o}\underset{\sim}{s}\, d\zeta_D \tag{6.42a}$$

On the other hand, if $\underset{\sim}{r} \neq \underset{\sim}{0}$, then eq. (6.41) shows readily that case (i) corresponds to <u>kinematic hardening</u>. This relation is in fact the equation of a hypershere in deviatoric stress space. It also represents the equation of a circle in <u>principal</u> stress space. In either case s_Y^o is the radius of the hypershere (circle) and $\underset{\sim}{r}$ is the radius vector which connects the origin of the stress space to the centre of the hypersphere (circle). The yield surface is, therefore, a translating spherical (circular) surface. Eq. (6.42) shows that the increment in plastic strain is normal to the yield surface. These results are shown diagrammatically in Figure 6. Note however that whereas in the classical theory of plasticity the concept of kinematic hardening was a conjecture, <u>here it is a derived result</u>.

Furthermore, we feel it important to emphasize that in the classical theory it is not known how the surface translates, i.e., it is not known

a priori how $\underset{\sim}{r}$ depends on the history of loading or plastic strain. For instance, Prager assumed that

$$d\underset{\sim}{r} = c\,d\underset{\sim}{\ell} \tag{6.43}$$

where c is a material constant. Ziegler suggested that

$$d\underset{\sim}{r} = d\mu\,(\underset{\sim}{s} - \underset{\sim}{r}) \tag{6.44}$$

where $d\mu$ is a positive quantity not specified. We point out that eq. (6.43) is equivalent to eq. (6.44) if $d\mu$ is proportional to $d\zeta_D$ as eq. (6.42) readily indicates. However, Prager's (or Ziegler's) rule is a particular case of the present theory as is pointed out in the following Remark.

Remark 1: Prager's rule of kinematic hardening is a particular case of the present theory obtained from eq. (6.37) but setting

$$\rho_1(\zeta_D) = \text{constant} \tag{6.45}$$

in which event,

$$\underset{\sim}{r} = 2\mu_o\rho_1\underset{\sim}{\ell} \tag{6.46}$$

This, as is well known, is called linear hardening in the sense that the stress strain curve in shear is linear beyond the onset of yield.

In this same vein, if $\rho_1(\zeta_D)$ is not a constant but consists of a single exponential, i.e.,

$$\rho_1(\zeta_D) = \rho_1 e^{-\alpha\zeta_D} \tag{6.47}$$

then it is easily shown as a result of eq.'s (6.37) and (6.38) that

$$d\underset{\sim}{r} = \frac{2\mu_o\rho_1}{s_Y^o}\,dz(\underset{\sim}{s} - \beta\underset{\sim}{r}) \tag{6.48}$$

where

$$\beta = 1 + \frac{\alpha s_Y^o}{2\mu_o\rho_1} \tag{6.49}$$

This is a new type of kinematic hardening. Note that the yield surface no longer translates along the outward normal at the extremity of the stress vector but in a direction which is skew. The skewness depends on the value of β.

Of course the above are particular cases. In general the translation vector is determined by eq. (6.37), in terms of a convolution product which involves the material function $\rho_1(\zeta_D)$. Within the assumptions of the present case, it is important to observe that $\rho_1(\zeta_D)$ can be determined by a simple shear experiment. In fact one can show readily from eq. (6.37) that

$$2\mu_o \rho_1(\zeta_D) = \frac{d\tau}{d\gamma_p}\bigg|_{\gamma_p = \zeta_D}, \tau > \tau_Y \tag{6.50}$$

where τ is the shear stress and γ_p the tensorial plastic shear strain component.

<u>Remark 2:</u> The function $\rho_1(\zeta_D)$ is determinate from a single monotonic shear test.

<u>Remark 3:</u> The mode of translation of the yield surface* is determined from a single monotonic shear experiment.

<u>Case (ii): $f(\zeta)$ monotonically increasing</u>

The counterparts of eq.'s (6.41) and (6.42) are now the following:

$$||\underset{\sim}{s} - \underset{\sim}{r}|| = f(\zeta_D) s_Y^o \tag{6.51}$$

$$d\underset{\sim}{\ell} = \frac{1}{s_Y^o f(\zeta_D)} \cdot (\underset{\sim}{s} - \underset{\sim}{r})\, d\zeta_D \tag{6.52}$$

*for all histories

Clearly, if $\underset{\sim}{r} = \underset{\sim}{0}$, then eq. (6.52) is that of a plastic material with isotropic hardening and a Von Mises yield criterion. In this case

$$d\underset{\sim}{Q} = \frac{1}{s_Y^o \, f(\zeta_D)} \underset{\sim}{s} \, d\zeta_D \tag{6.52a}$$

On the other hand if $\underset{\sim}{r} \neq \underset{\sim}{0}$ then eq. (6.52) shows that the yield surface now expands as well as translates. The increment of plastic strain is still normal for the yield surface. The translation vector $\underset{\sim}{r}$ is still given by eq. (6.37). We shall not go into the details of determining $f(\zeta)$ and $\rho_1(\zeta_D)$ in this case, at this juncture. The reader is referred to Appendix D.

The constitutive equation for $\sigma > \sigma_Y$ for the hydrostatic response is identical in form to the above and can be written down by inspection using analogous terminology. To wit, when $\kappa_o = 1$, then:

$$\sigma = \sigma_Y^o \frac{d\theta}{dz_H} + K_o \int_o^{z_H} \phi_1(z_H - z_H') \frac{\partial\theta}{dz_H'} \, dz_H' \tag{6.53}$$

where $\sigma = \sigma_{kk}/3$. If $\sigma \leq \sigma_Y$, then the response is elastic and

$$\sigma = K_o \, \varepsilon_{kk} \quad . \tag{6.54}$$

Of course, the above discussion applies only to the case where k_o is equal to unity.

7. Elastic Response as Points of Interdeterminancy of $\frac{d\mathfrak{Q}}{dZ_D}$.

In this Section we shall show that when $k_1 = 1$, there exist physical processes for which $d\zeta_D$ is equal to zero and that in fact these processes are associated with elastic deformation. Analogous conclusions can be drawn with regard to $d\zeta_H$ when $k_o = 1$. The above remarks are made in the context of assumed strict independence between deviatoric and hydrostatic response, in the sense that a history of deviatoric strain has no effect on the hydrostatic response and correspondingly a history of hydrostatic strain has no effect on the deviatoric response.*

It is important for our purposes to introduce certain definitions:

Deviatoric plastic strain space: The space of deviatoric strain components with metric δ_{ij}. The coordinates of this space will be denoted by χ_i, $(i = 1, 2, \ldots, 9)$ corresponding to the components of the tensor $\mathfrak{Q}_{ij}$.

Deviatoric plastic strain path: A continuous line in χ_i space.

Remark: The path determines the variation of χ_i in this space. We limit the discussion to paths that pass through the origin. In this event $\chi_i = 0$ at some point in the path, in fact the origin.

Let the extremity of the vector $\vec{\chi}$ change continuously along the path. We define quantities s and ζ as follows:

s: The distance of the point χ_i from the origin measured along the

*Henceforth the suffix D will be omitted. Also, this chapter will apply exclusively to deviatoric response and occasional omission of the word "deviatoric" must not be construed to imply otherwise.

path, where χ_i is the current position of the extremity of the vector $\vec{\chi}$.

ζ: The cummulative distance travelled by the extremity of $\vec{\chi}$, i.e.

$$d\zeta = |ds| \tag{7.1}$$

or

$$\zeta = \int |ds| \tag{7.2}$$

<u>Stipulations</u>:

There exists a point on the path at which s = 0; in the present case this is the origin of the deviatoric strain space.

There exists a positive direction of s, which is the direction of increasing s along the path. In a similar sense there exists a negative direction of s which is the direction of decreasing s along the path.

The path is separated into two regions of negative and positive s with a common point which is the origin of χ_i; s is equal to zero at this point.

In a physical context we note that in the case of simple shear, for instance, the shear strain is positive when the material is sheared in a specific (positive) direction and negative when shearing takes place in the opposite direction. In a similar sense in a uniaxial plastic strain experiment there exists a positive (tensile) strain and a negative (compressive) strain. Of course this is precisely the meaning of s in one dimensional tests.

<u>Remark I</u>. Given a path, the derivative $\frac{ds}{d\zeta}$ experiences a jump equal to 2 when one traverses the origin in the positive s direction. The jump is equal to - 2 when the origin is traversed in the negative s direction.

The proof is trivial: $\frac{ds}{d\zeta} = 1$ at $s = 0+$; $\frac{ds}{d\zeta} = -1$ at $s = 0-$.

Remark II. The derivative $\frac{ds}{d\zeta}$ is indeterminate at $s = \zeta = 0$.

Figure 7 illustrates the above statements.

Discussion of eq. (6.36) at $\zeta = 0$.

At $\zeta = 0$, and therefore at $s = 0$

$$\underset{\sim}{s} = s_Y^o \frac{d\underset{\sim}{\mathcal{Q}}}{dz} \tag{7.3}$$

Therefore,

$$\underset{\sim}{s} = s_Y^o \frac{d\underset{\sim}{\mathcal{Q}}}{d\zeta} \tag{7.4}$$

since at $\zeta = 0$, $f(o) = 1$.

On the basis of eq. (7.4)

$$||\underset{\sim}{s}||^2 = (s_Y \frac{ds}{d\zeta})^2 \tag{7.5}$$

But the right hand side of eq. (7.5) is indeterminate at $\zeta = 0$, hence $||\underset{\sim}{s}||$, and therefore $\underset{\sim}{s}$, are also indeterminate. However $\zeta = 0$, requires that

$$\underset{\sim}{\mathcal{Q}} = \underset{\sim}{0} \tag{7.6}$$

Equation (7.6), therefore, and the definition of $\underset{\sim}{\mathcal{Q}}$ (eq. (6.5) with $k_1 = 1$) lead to the relation,

$$\underset{\sim}{s} = 2\mu_o \underset{\sim}{e} \tag{7.7}$$

which is a statement to the fact that at $\zeta = 0$, the deviatoric response is elastic.

Equation (6.36) at ζ = 0+ and 0-.

In the vicinity of s = 0, let $\underset{\sim}{\ell}$ be given by the relation

$$\underset{\sim}{\ell} = \underset{\sim}{\ell} s \tag{7.8}$$

where ℓ_{ij} are the direction cosines at s = 0. Then as a result of eq. (6.36) (note that the second term of the right hand side of eq. (6.36) tends to zero at z → 0)

$$\underset{\sim}{s}_+ = s_Y^o \ell_{ij} \tag{7.9a}$$

$$\underset{\sim}{s}_- = -s_Y^o \ell_{ij} \tag{7.9b}$$

where $\underset{\sim}{s}_+ = \underset{\sim}{s}(0+)$ and $\underset{\sim}{s}_- = \underset{\sim}{s}(0-)$. The deviatoric stress tensor is therefore discontinuous, with a discontinuity of magnitude $2s_Y^o$, at s = ζ = 0.

Equation (6.36) at Reversal Points.

Definition: A reversal point R is a point on the strain path s at which there is a reversal in the sign of the strain increment. In effect at R

$$d\chi_i^- = - d\chi_i^+ \tag{7.10}$$

With reference to Figure 8, the following relations are of interest:

$$\frac{d\chi_i}{ds}(R') = \frac{d\chi_i}{ds}(R) + \theta_i \tag{7.11}$$

$$\text{Lim } \theta_i = 0, \quad R' \to R.$$

However, as a result of our definitions of ds and $d\zeta$:

$$\lim_{R' \to R} \frac{ds}{d\zeta}\Big|_{R'} = 1 \tag{7.12a}$$

$$\lim_{R'' \to R} \frac{ds}{d\zeta}\Big|_{R''} = -1 \tag{7.12b}$$

Hence, invoking eq. (6.10):

$$\lim_{R' \to R} \frac{ds}{dz}\Big|_{R'} = f(\zeta)\Big|_{R} \tag{7.13a}$$

$$\lim_{R'' \to R} \frac{ds}{dz}\Big|_{R''} = -\,f(\zeta)\Big|_{R} \tag{7.13b}$$

Also,

$$\lim_{R' \to R} \underset{\sim}{r}(R') = \underset{\sim}{r}(R) \tag{7.14a}$$

$$\lim_{R'' \to R} \underset{\sim}{r}(R'') = \underset{\sim}{r}(R). \tag{7.14b}$$

Let in the vicinity of R

$$d\underset{\sim}{\mathcal{R}} = \underset{\sim}{\ell}\,ds \tag{7.15}$$

where ℓ_{ij} are the direction cosines of the tangent to the path at R. Then as a result of eq.'s (7.11) through (7.15) and eq. (6.36):

$$\underset{\sim}{s}(R') = s_Y^o \ell_{ij} f_R(\zeta) + \underset{\sim}{r}(R), \tag{7.16a}$$

$$\underset{\sim}{s}(R'') = - s_Y^o \ell_{ij} f_R(\zeta) + \underset{\sim}{r}(R), \tag{7.16b}$$

in the limits $R' \to R$, $R'' \to R$.

Since in fact ℓ_{ij} is normal to the stress hypersphere at R and $s_Y^o f(\zeta)$ is the current radius of the hypersphere, eq.'s (7.16a,b) admit the geometric construction shown in Figure 9.

Note that the response in the stress range $R'R''$ is elastic, and corresponds to constant ζ and z. During the elastic response $d\zeta$ is therefore zero and the constitutive equation is simply

$$d\underset{\sim}{s} = 2\mu_o d\underset{\sim}{e} \tag{7.17}$$

A plot of ζ versus s for this process is shown in Figure 10.

Clearly, at R, the slope $\frac{ds}{d\zeta}$ is indeterminate.

<u>More General Deformation Processes With Constant ζ.</u>

We begin with eq.'s (6.36) and (6.37) which we write in the form

$$\underset{\sim}{s} = s_Y^o \frac{d\underset{\sim}{Q}}{dz} + \underset{\sim}{r} \tag{7.18}$$

or, equivalently,

$$\underset{\sim}{s} = s_Y^o \int_o^z \delta(z - z') \frac{d\underset{\sim}{Q}}{dz'} dz' + \underset{\sim}{r} \tag{7.19}$$

Since, however, the delta function in eq. (7.19) was obtained by a limiting process in which $k_1 \to 1$, we replace $\delta(z)$ by a function which in the above limiting process becomes a delta function, i.e., we write

$$\underset{\sim}{s} = \frac{2\mu_o}{1-k_1} \int_o^z e^{-\alpha(z - z')} \frac{d\underset{\sim}{\mathfrak{z}}}{dz'} dz' + \underset{\sim}{r}(z) \tag{7.20}$$

where

$$\alpha = \frac{2\mu_o}{(1-k_1)s_Y^o} \tag{7.21}$$

and $\underset{\sim}{s}$ is obtained from eq. (7.20) by a limiting process in which k_1 tends to unity. Equation (7.20) may now be written as a differential equation in the form:

$$\frac{1}{s_Y^o}(\underset{\sim}{s} - \underset{\sim}{r})dz + \frac{1-k_1}{2\mu_o} d(\underset{\sim}{s} - \underset{\sim}{r}) = d\underset{\sim}{\mathfrak{z}} \ . \tag{7.22}$$

We note in passing that in the limit $k_1 \to 1$

$$||\underset{\sim}{s} - \underset{\sim}{r}|| = s_Y^o f(\zeta) = s_Y \tag{7.23a, b}$$

a result given previously in eq. (6.41). We also recall the definition of $d\underset{\sim}{\mathfrak{z}}$:

$$d\underset{\sim}{\mathfrak{z}} = d\underset{\sim}{e} - \frac{k_1}{2\mu_o} d\underset{\sim}{s} \tag{7.24}$$

Equations (7.22) and (7.24) combine to give the result

$$k_1 \frac{\hat{s}}{s_Y} d\zeta + (1-k_1)d\hat{\underset{\sim}{e}} = d\underset{\sim}{\mathfrak{z}} \tag{7.25}$$

where

$$\hat{\underset{\sim}{s}} = \underset{\sim}{s} - \underset{\sim}{r} \tag{7.26}$$

$$d\hat{\underset{\sim}{e}} = d\underset{\sim}{e} - \frac{k_1}{2\mu_o} d\underset{\sim}{r} \tag{7.27}$$

However, from eq (6.37)

$$d\underset{\sim}{r} = 2\mu_o(\rho_1(o)d\underset{\sim}{\ell} + \underset{\sim}{h}\ dz) \tag{7.28}$$

where

$$\underset{\sim}{h} = \int_o^z \rho_1'(z - z') \frac{d\underset{\sim}{\ell}}{dz'} dz' \tag{7.29}$$

and

$$\rho_1' = \frac{d\rho_1}{dz} \quad (0 < z < \infty) \tag{7.30}$$

Thus, ρ' is well behaved (continuous, monotonically decreasing) function of z. Note that $\underset{\sim}{h}(0) = \underset{\sim}{0}$.

Equations (7.25) and (7.28) combine to give the following equation for dz:

$$\underset{\sim}{a}dz + (1-k_1)d\underset{\sim}{e} = cd\underset{\sim}{\ell} \tag{7.31}$$

where

$$\underset{\sim}{a} = \{k_1 \frac{\hat{\underset{\sim}{s}}}{s_Y^o} - k_1(1 - k_1)\underset{\sim}{h}\}/f(\zeta) \tag{7.32}$$

$$c = 1 + k_1(1 - k_1)\rho_1(o) \tag{7.33}$$

By taking the norm of both sides of eq. (7.31) one obtains the following

quadratic equation in $d\zeta$

$$\frac{(c^2 - ||\underset{\sim}{a}||^2)}{1 - k_1} d\zeta^2 - 2\, \underset{\sim}{a} \cdot \underset{\sim}{de}\, d\zeta - (1-k_1)||\underset{\sim}{de}||^2 = 0 \quad (7.34)$$

The two roots of eq. (7.34) are given by eq. (7.35):

$$d\zeta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{de} \pm \sqrt{\underset{\sim}{a} \cdot \underset{\sim}{de} + (1-k_1)C||\underset{\sim}{de}||^2}}{C} \quad (7.35)$$

where

$$C = \frac{c^2 - ||a||^2}{1 - k_1} \quad (7.36)$$

It may be shown that

$$\lim_{k_1 \to 1} C = C_1 \quad (7.37)$$

where

$$C_1 = 2(1 + \rho_1(o) + \frac{\underset{\sim}{\hat{s}} \cdot \underset{\sim}{h}}{s_Y^o f(\zeta)^2}) \quad (7.38)$$

In the limit of $k_1 \to 1$, the two roots of $d\zeta$ are, therefore:

$$d\zeta = \begin{cases} \underset{\sim}{\hat{s}} \cdot \underset{\sim}{de}/C_1 \\ 0 \end{cases} \quad (7.39)$$

with the constraint $d\zeta \geqslant 0$.

The Two Roots of $d\zeta$.

At the initiation of plastic deformation $\underset{\sim}{h}$ is equal to zero as evidenced by eq. (7.29). Now, from eq.'s (6.29) and (6.30)

$$\rho_1(\zeta) = \sum_{r=1}^{n-1} R_r e^{-\beta_r \zeta} \tag{7.40}$$

where R_r are all positive as shown in Appendix B. Therefore, $\rho_1(o)$ is positive. However, the scalar product $\hat{\underset{\sim}{s}}\cdot\underset{\sim}{h}$ may be negative. In fact, in the particular case where $\rho_1(\zeta)$ consists of a single exponential term, i.e.

$$\rho_1(\zeta) = +R_1 e^{-\beta_1 \zeta} \tag{7.41}$$

The function $\rho_1'(\zeta)$ is given by eq. (7.42):

$$\rho_1'(\zeta) = -\beta_1 R_1 e^{-\beta\zeta} \tag{7.42}$$

in which event

$$\underset{\sim}{h}(\zeta) = -\beta_1 \frac{\underset{\sim}{r}(\zeta)}{2\mu_o} \quad .$$

Thus

$$\hat{\underset{\sim}{s}}\cdot\underset{\sim}{h} = -\frac{\beta_1}{2\mu_o}\hat{\underset{\sim}{s}}\cdot\underset{\sim}{r} \tag{7.43}$$

The scalar product $\hat{\underset{\sim}{s}}\cdot\underset{\sim}{h}$ will, therefore, be negative if the angle between $\hat{\underset{\sim}{s}}$ and $\underset{\sim}{r}$ is acute --in this particular case. It is therefore quite possible that during the process of deformation C_1 will decrease from a positive value to zero.

We shall first investigate the situation where C_1 is bounded by the inequalities

$$0 < C_1 < 2\{1 + \rho_1(0)\}$$

Case (i): $\hat{\underset{\sim}{s}} \cdot d\underset{\sim}{e} > 0$

In this case the dissipation is positive, see eq. (2.14), since $d\zeta$ is in fact positive and is given by the eq. (7.44)

$$d\zeta = \frac{\hat{\underset{\sim}{s}} \cdot d\underset{\sim}{e}}{C_1} \tag{7.44}$$

The process is admissible.

Case (ii): $\hat{\underset{\sim}{s}} \cdot d\underset{\sim}{e} < 0$

In this case the dissipation is negative (since $d\zeta$ is negative, see eq. (2.14)) and the process is inadmissible. Therefore the second root of $d\zeta$ must be chosen and as a result:

$$d\zeta = 0. \tag{7.45}$$

In this event the dissipation is zero and the deformation is elastic since eq. (7.45) implies that

$$d\underset{\sim}{q} = 0 \tag{7.46}$$

in which case

$$d\underset{\sim}{s} = 2\mu_o d\underset{\sim}{e}. \tag{7.47}$$

<u>Remark</u>: Case (ii) is the "unloading condition" given unequivocably by the endochronic theory. More specifically, given a state of stress on the

yield surface, the process is dissipative and $d\zeta > 0$, if the strain increment makes an acute angle with the radius vector through the point which represents the state of stress; the process is elastic if the angle is equal to or greater than ninety degrees.

The Case: $C_1 = 0$

The situation where $C_1 \to 0$ results in $d\zeta$ becoming unboundedly large in which event the material flows without limit. This event will occur when

$$\hat{\underset{\sim}{s}} \cdot \underset{\sim}{h} = - s_Y^o(1 + \rho_1(o))f(\zeta)^2 \tag{7.48}$$

Stress Paths Within the Yield Surface.

If eq. (6.36) is to apply, $d\zeta$ must be different from zero (in fact positive), since otherwise the derivative $\frac{d\underset{\sim}{s}}{d\zeta}$ is not determinate. Therefore, if eq. (6.36) applies, then the incremental process emanating from state $\underset{\sim}{s}$ is dissipative and vice-versa. In the same vein, eq. (6.51) is a direct consequence of eq. (6.36). Therefore if an incremental process originating at state $\underset{\sim}{s}$ is to be dissipative eq. (6.51) must necessarily apply. It follows as a corollary that stress states which do not satisfy eq. (6.51) cannot lead immediately to dissipative processes.

Remark: State $\underset{\sim}{s}$ which lie within the yield surface cannot lead immediately to dissipative incremental processes.

It follows from the above remark that all incremental processes which emanate from stress states within the yield surface are elastic ($d\zeta = 0$), in which event for all $d\underset{\sim}{s}$ emanating from such state $\underset{\sim}{s}$:

$$d\underset{\sim}{s} = 2\mu d\underset{\sim}{e} \tag{7.49}$$

Incremental Determination of the Constitutive Response.

Given that the constitutive properties of a material are known, let, at some point in the course of the deformation (strain) history, $\hat{\underset{\sim}{s}}$ and $\underset{\sim}{h}$ also be known. Then C_1 can be determined from eq. (7.38). Given an increment of strain $d\underset{\sim}{e}$, eq. (7.39) determines $d\zeta$, depending on the sign of $\hat{\underset{\sim}{s}} \cdot d\underset{\sim}{e}$. If the sign is negative or zero, then $d\zeta = 0$ and $d\underset{\sim}{s}$ is given by eq. (7.47). If the sign is positive then

$$d\zeta = \frac{\hat{\underset{\sim}{s}} \cdot d\underset{\sim}{e}}{C_1} \qquad (7.50)$$

and

$$dz = \frac{d\zeta}{f(\zeta)} \qquad (7.51)$$

In the limit of $k_1 \to 1$, then as a result of eq. (7.22) or eq. (7.18)

$$d\underset{\sim}{Q} = \frac{\hat{s}_o}{s_Y} dz \qquad (7.52)$$

Hence, $d\underset{\sim}{Q}$ may be calculated. Knowledge of $d\underset{\sim}{Q}$ leads to the direct determination of $d\underset{\sim}{s}$ from eq. (7.24) in the limit of $k_1 \to 1$, i.e.,

$$d\underset{\sim}{s} = 2\mu_o(d\underset{\sim}{e} - d\underset{\sim}{Q}) \qquad (7.53)$$

Clearly $d\underset{\sim}{r}$ may now be found from eq. (7.28), and the procedure may thus be repeated at will.

We note that at the initiation of plastic deformation $\underset{\sim}{r} = \underset{\sim}{0}$, $\underset{\sim}{h} = \underset{\sim}{0}$ and $\underset{\sim}{s} = 2\mu_o\underset{\sim}{e}$.

8. The Closure of Hysteresis Loops.

The question of closure is treated by means of an application to aluminum undergoing stress reversals in a uniaxial stress field. One has difficulties in obtaining closed hysteresis loops in the first quadrant of the stress-strain diagram, if one uses the definition of intrinsic time of Reference 1, i.e., eq. (1.1). The purpose of this Section is to demonstrate that no such difficulties arise when

$$d\zeta = |d\theta_1| \tag{8.1}$$

and

$$\theta_1 = \varepsilon_1 - \frac{k\sigma_1}{E} \tag{8.2}$$

and $k = 1$.

For the purpose of demonstration we use the constitutive equation

$$\sigma_1 = \sigma_Y \frac{d\theta_1}{d\zeta} + \int_0^{\zeta} E(\zeta-\zeta') \frac{d\theta_1}{d\zeta'} \, d\zeta' \tag{8.3}$$

where

$$E(\zeta) = E_1 \, e^{-\alpha\zeta} + E_2 \tag{8.4}$$

with the following values of the constants: $\sigma_Y = 5$ tons/in^2, $E_1 = 1500$ tons/in^2, $E_2 = 200$ tons/in^2, $\alpha = 500$. The elastic modulus E_o is equal to 4.46×10^3 tons/in^2; however, this value is not pertinent to the present application since σ_1 is plotted directly against θ_1. The computation of the stress response to loading-unloading-reloading histories is straightforward and will not be treated in detail here. For a more elaborate treatment of this problem the reader is referred to Reference 2. The results are shown in Figure 11.

The point A is the initial yield event. Point B denotes the point of unloading and C, D, E, F, G, and H are points of reloading. Note that points such as C, where $\sigma_C \geq \sigma_B - 2\sigma_Y$, gives rise to reloading paths which coincide with the unloading paths. In such a case the area of the hysteresis loops is zero. However, reloading from points where $\sigma < \sigma_B - 2\sigma_Y$ gives rise to loops that are closed, as shown in the Figure. In this constitutive theory as it is constituted by equations (7.1), (7.2) and (7.3), no open loops exist for $k = 1$.

9. Discussion

In this paper we have presented an endochronic theory of plasticity which is based on a new definition of intrinsic time which is broader than the one we proposed before.[1] In its two limiting forms the intrinsic time measure is expressible either in terms of the increment of the total deformation on one hand, or in terms of the increment of the plastic deformation, on the other. An intermediate definition is made possible by means of a material tensor $\phi_{ijk\ell}$.

The theory is shown to predict essential features of the plastic response of metals and in certain asymptotic cases it evinces commonalities with classical theories of plasticity and contains a number of classical plasticity models as special cases. In particular, the existence of a yield surface is shown as a consequence, when the intrinsic time measure is defined in terms of the increment of plastic strain. Also, the response of an elastic perfectly plastic solid is shown to be predictable by the theory. However, the most significant feature of the present theory is that it predicts correctly--in the case of metals where k_1 is almost equal to unity--the stress response to strain histories involving reversals in the sign of the strain rate. This

is a particularly powerful result, as one and the same constitutive equation is used in the prediction of the stress response to such reversals. When k is equal to unity a dichotomy in the constitutive representation results in the sense that an elastic constitutive relation must be used in the course of strain histories involving "unloading".

It is felt that the endochronic theory has now evolved to a stage where it can be used in design as a predictive tool in the understanding of the mechanical response of metals and other materials.

Appendix A

We solve eq. (6.19) by the Laplace Transform technique, where

$$G(z) = G_1 e^{-\alpha_1 z} + G_2 e^{-\alpha_2 z} \tag{A.1}$$

Then, as a result of eq. (6.19),

$$\bar{\rho} = \frac{p + \bar{\alpha}}{(1-k_1)\, p^2 + (\alpha_1 + \alpha_2 - k\bar{\alpha})\, p + \alpha_1\alpha_2} \tag{A.2}$$

where

$$\bar{\alpha} = G_2\alpha_1 + G_1\alpha_2 \quad . \tag{A.3}$$

It can be shown by straightforward analysis that as $k_1 \to 1$

$$\bar{\rho} \to \frac{p + \bar{\alpha}}{(1-k_1)(p+\beta_1)(p+\beta_2)} \tag{A.4}$$

where

$$\beta_1 = \frac{1}{1-k_1} (\alpha_1 G_1 + \alpha_2 G_2) \tag{A.5}$$

$$\beta_2 = \frac{\alpha_1\alpha_2}{\alpha_1 G_1 + \alpha_2 G_2} \tag{A.6}$$

In the limit of $k_1 \to 1$ and as a result of eq. (A.4), $\rho(z)$ may be written explicitly in the form:

$$\rho(z) = \frac{1}{1-k_1} \exp\left(-\frac{\lambda}{1-k_1} z\right) + G_1 G_2 \left(\frac{\alpha_2-\alpha_1}{\lambda}\right)^2 \exp\left(-\frac{\alpha_1\alpha_2}{\lambda} z\right). \tag{A.7}$$

where $\lambda = G_1\alpha_1 + G_2\alpha_2$. (A.8)

It is worth repeating that as $k_1 \to 1$, the first term on the right-hand side of eq. (A.7) becomes proportional to a delta function and

$$\rho(z) = \frac{1}{\lambda}\,\delta(z) + G_1G_2\left(\frac{\alpha_2-\alpha_1}{\lambda}\right)^2 \exp\left(-\frac{\alpha_1\alpha_2}{\lambda} z\right). \tag{A.9}$$

The fact that $\delta(z)$ is an essential part of $\rho(z)$ when $k_1 = 1$, was demonstrated in the text in the case of any number of internal variables.

Appendix B

We note the following relations:

$$\bar{R} = \bar{Q} - p\bar{P} \tag{B-1}$$

$$\bar{P} = \bar{Q} \sum_r \frac{G_r}{p + \alpha_r} \tag{B-2}$$

We also note that, since $-\beta_s$ are the zeros of $\bar{R}$, it follows from eq. (B-1) that

$$\bar{Q}(-\beta_s) = -\beta_s \bar{P}(-\beta_s) \tag{B-3}$$

Also from eq. (B-2):

$$\bar{P}(-\beta_s) = \bar{Q}(-\beta_s) \sum_r \frac{G_r}{-\beta_s + \alpha_r} \tag{B-4}$$

It follows from eq.'s (B-3) and (B-4) that

$$\sum_r \frac{G_r}{-\beta_s + \alpha_r} = -\frac{1}{\beta_s} \tag{B-5}$$

Also from eq.'s (B-1) and (B-2)

$$1 - \sum_r \frac{pG_r}{p + \alpha_r}\bigg|_{p = -\beta_s} = 0 \tag{B-6}$$

But, from eq. (6.22),

$$\sum G_r = 1 \tag{B-7}$$

Hence from eq.'s (B-6) and (B-7)

$$\sum_r \frac{\alpha_r G_r}{-\beta_s + \alpha_r} = 1 \tag{B-8}$$

At this point we are in a position to evaluate $\overline{R}'$. To this end we note that as a result of eq. (6.24),

$$\overline{Q}' = \overline{Q} \sum_r \frac{G_r}{p + \alpha_r} \tag{B-9}$$

Also in view of eq. (B-2):

$$P' = \overline{Q} \sum_r \frac{1}{p + \alpha_r} \sum_r \frac{G_r}{p + \alpha_r} - \overline{Q} \sum_r \frac{G_r}{(p + \alpha_r)^2} \tag{B-10}$$

Therefore it follows from eq. (B-1) and eq.'s (B-9) and (B-10) that,

$$\overline{R}' = \overline{Q}\sum_r \frac{1}{p + \alpha_r} - \overline{Q}\sum_r \frac{G_r}{p + \alpha_r} - p \left\{ \overline{Q}\sum_r \frac{1}{p + \alpha_r} \sum_r \frac{G_r}{p + \alpha_r} \right.$$

$$\left. - \overline{Q}\sum_r \frac{G_r}{(p + \alpha_r)^2} \right\} \tag{B-11}$$

At this point evaluating $\overline{R}'$ at $-\beta_s$ and invoking eq. (B-6), we find that

$$\overline{R}'(-\beta_s) = -\overline{Q}(-\beta_s) \left[\sum_r \frac{G_r}{-\beta_s + \alpha_r} + \sum_r \frac{G_r \beta_s}{(-\beta_s + \alpha_r)^2} \right] \tag{B-12}$$

or

$$\overline{R}'(-\beta_s) = -\overline{Q}(-\beta_s) \sum_r \frac{\alpha_r G_r}{(\alpha_r - \beta_s)^2} \tag{B-13}$$

Now from (B-1)

$$\overline{P}(-\beta_s) = \frac{\overline{Q}(-\beta_s)}{-(\beta_s)} \tag{B-14}$$

At this point use of eq. (6.28) gives an explicit expression for R_s given by eq. (B-15)

$$R_s = \frac{1}{\beta_s \sum_r \frac{\alpha_r G_r}{(\alpha_r - \beta_s)^2}} \tag{B-15}$$

However, the constants α_r, β_r and G_r are all positive. Thus the residues R_s are all <u>positive</u>!

Appendix C

In the theory of Laplace Transform, one may deal with functions with simple jump discontinuities by enlarging the class of differentiable functions to include such functions. This is achieved by making the Heavinside step function a member of this class and by stipulating that the derivative of this function is the Dirac delta function, i.e.,

$$\frac{d}{dt} H(t) = \delta(t) \tag{C-1}$$

If one does not do this one can only deal with differentiable functions in the classical sense in which case if such a function $f(t)$ is such that $f(0) \neq 0$, then $f(t)$ is defined in the interval $0 < t < \infty$ and, of course, as is well known

$$L\left(\frac{df}{dt}\right) = p\bar{f}(p) - f(0+). \tag{C-2}$$

where L denotes the Laplace transform over the open interval $(0, \infty)$ with respect to the parameter p.

It follows from (C-2) that

$$p\bar{f}(p) = f(0+) + L\left(\frac{df}{dt}\right) \tag{C-3}$$

or

$$L^{-1}p\bar{f}(p) = f(0+)\,\delta(t) + \left.\frac{df}{dt}\right|_{t>0} \tag{C-4}$$

If, however, functions with simple jump discontinuities are admitted, the Laplace transformation may be used over the half open interval $[0_1 \infty)$ and a function $f_o(t)$ can now be defined on this interval with the condition that $f_o(0) = 0$, $f_o(t) = f(t)$, $t > 0$. This is what we have done in this paper. This function $f_o(t)$ now has a jump discontinuity at $t = 0$ of strength $f(0+)$.

In this case

$$L\left(\frac{df_o}{dt}\right) = p\bar{f}_o(p) \tag{C-5}$$

Note that $\bar{f}_o(p) = \bar{f}(p)$ since f_o differs from f only on a set of measure zero.

Thus, using eq. (C-5)

$$L\,\frac{df_o}{dt} = p\bar{f}(p)\ . \tag{C-6}$$

since $f_o(o) = 0$.

It follows that

$$L^{-1}p\bar{f}(p) = \frac{df_o}{dt} = f(0+)\,\delta(t) + \frac{df}{dt}\Big|_{t>0} \tag{C-7}$$

in accordance with eq. (C-1).

Appendix D

Experimental determination of the functions $\rho_1(z)$ and $f(\zeta)$.

In pure shear eq. (6.36) becomes

$$\tau = s_Y^o \frac{d\gamma^p}{dz} + 2\mu_o \int_o^z \rho_1(z - z') \frac{d\gamma^p}{dz'} dz' \tag{D-1}$$

where τ is the shear stress and γ^p the plastic shear strain. We recall the relation

$$dz = \frac{d\zeta}{f(\zeta)} \tag{D-2}$$

The suffix D on ζ and z has been omitted for simplicity of notation.

With reference to Figure 12 which is self-explanatory and eq. (D-1), the following results are evident:

(i) ζ is the plastic shear strain at point A, and is equal to the distance oA′.

(ii) The equation below holds

$$\tau_A - \tau_{Y'} = s_Y^o f(\zeta_A) \tag{D-3}$$

Thus $f(\zeta)$ can be determined from measurements of the yield points upon loading (point Y) and load reversal (point Y′). The function $z(\zeta)$ may now be determined by integration of eq. (D-2). Thus we have the functional relation

$$f(\zeta) = f\{\zeta(z)\} = \hat{f}(z) \tag{D-4}$$

At this juncture we show that $\rho_1(z)$ can be determined from a simple monotonic loading test. During such a test dz is related to $d\gamma$ by eq. (D-5)

$$dz = \frac{d\gamma^p}{f(\gamma^p)} \tag{D-5}$$

since, in fact, during loading

$$d\zeta = d\gamma^P \tag{D-6}$$

and

$$\zeta = \gamma^P \tag{D-7}$$

upon observation of the condition $\zeta = 0$ when $\gamma = 0$. As a result of eq.'s (D-4) through (D-7), eq. (D-1) reduces to the following form:

$$\tau = s_Y^o \hat{f}(z) + 2\mu_o \int_o^z \rho_1(z - z')\hat{f}(z')dz' \tag{D-8}$$

However since $\tau(\gamma^P)$ is known, then so is $\tau(\zeta)$ and hence $\hat{\tau}(z) = \tau\{\zeta(z)\}$. Thus, since $\hat{f}(z)$ is also known, the function $\hat{g}(z)$, where

$$\hat{g}(z) = \hat{\tau}(z) - s_Y\hat{f}(z) \tag{D-9}$$

is a known function of z. Equation (D-8) therefore, now reduces to the following Volterra integral equation

$$2\mu_o \int_o^z \hat{f}(z - z')\, \rho_1(z')dz' = \hat{g}(z) \tag{D-10}$$

where the right hand side is a known function.

Equation (D-10) determines $\rho_1(z)$ uniquely if $\hat{f}$ is a continuous bounded function of z in the interval (0,z). The function $\hat{f}$ is expected to have these properties on physical grounds.

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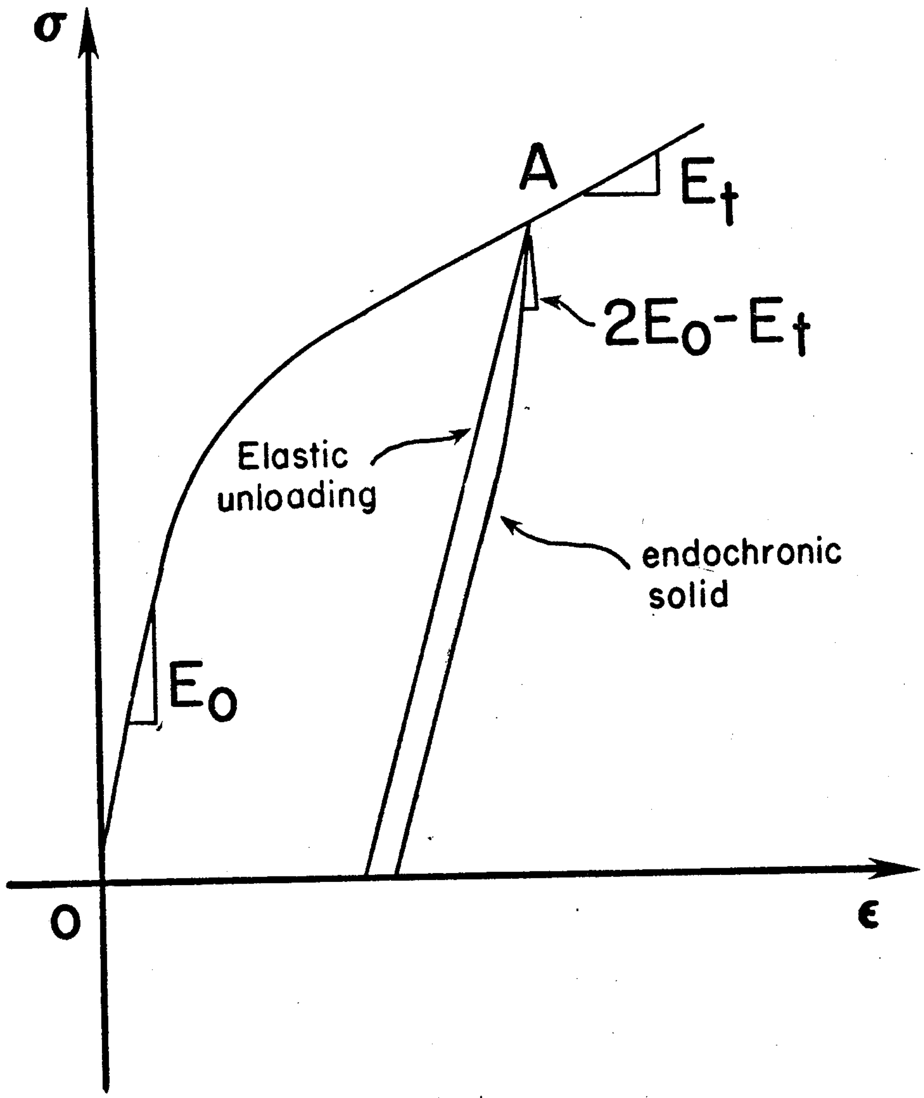

Fig. 1. Unloading response of an endochronic solid.

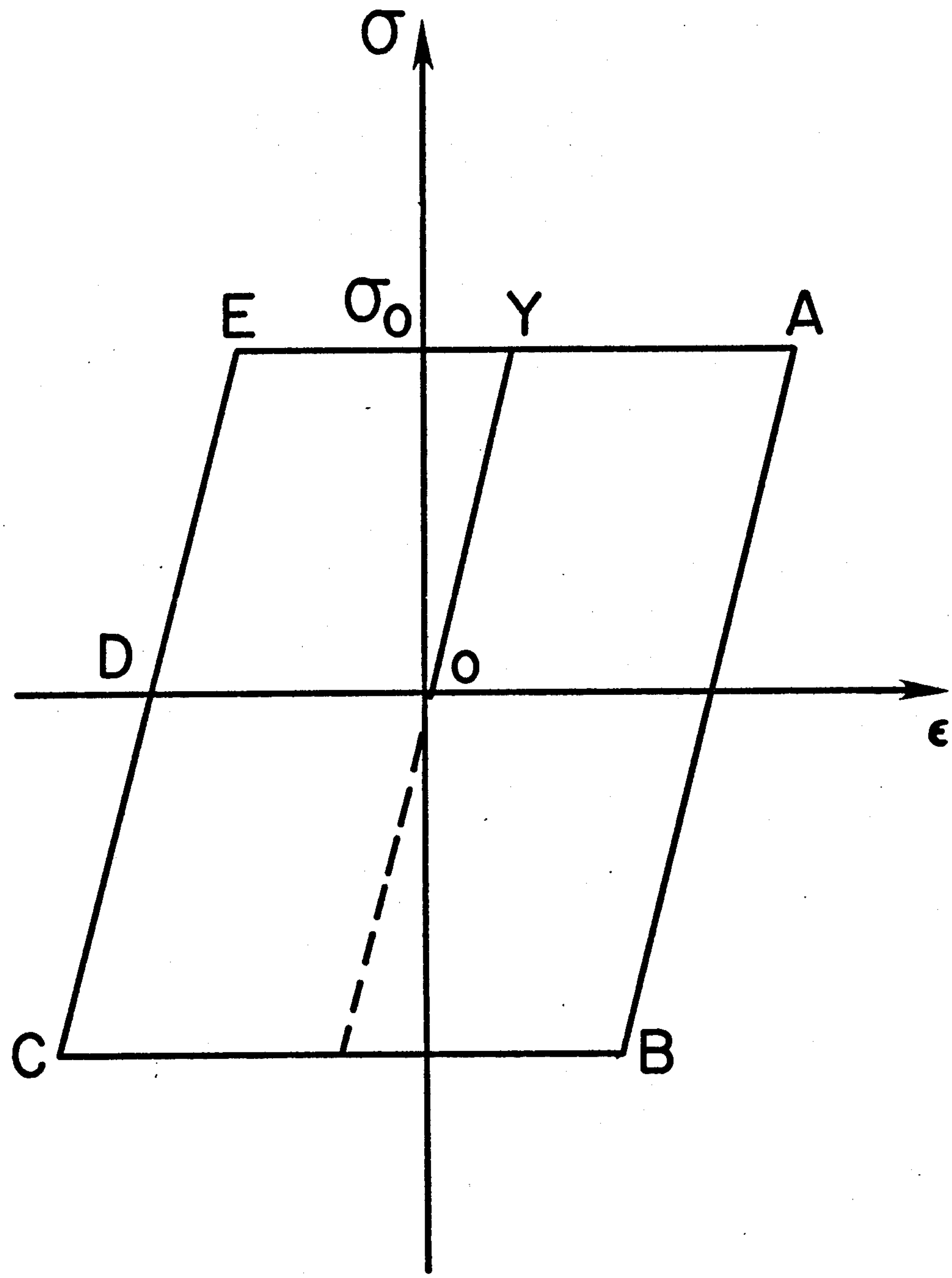

Fig. 2. An elastic-perfectly plastic solid.

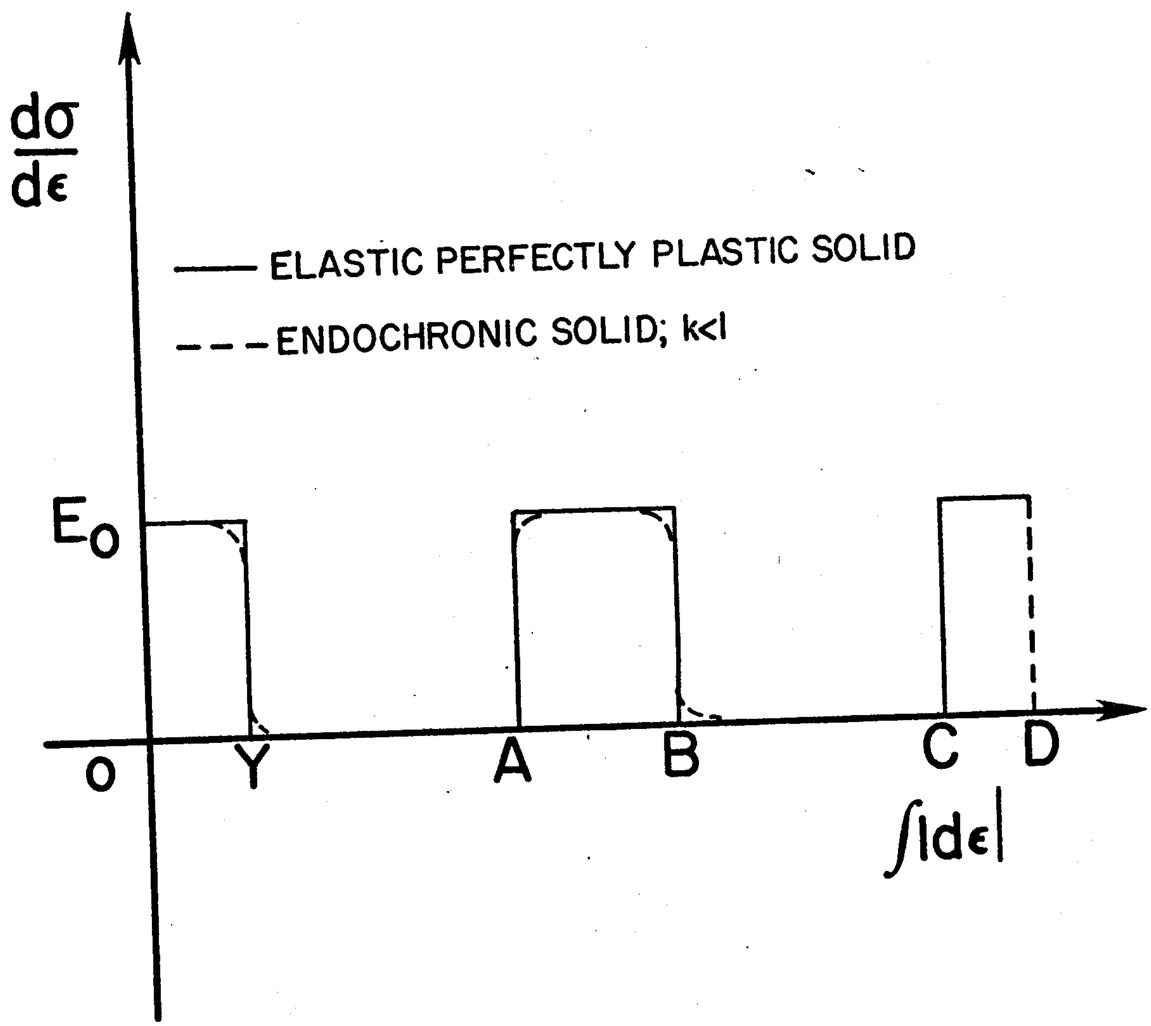

Fig. 3. Comparison of an elastic-perfectly plastic and an endochronic solid.

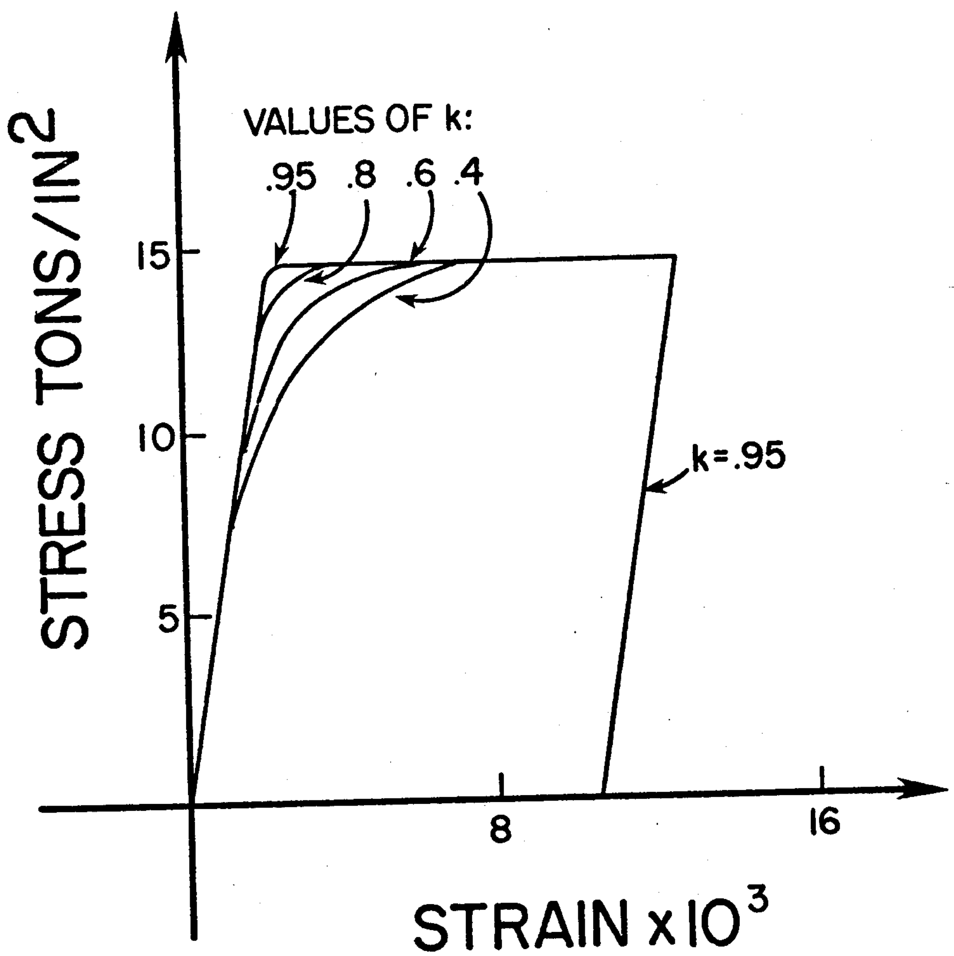

Fig. 4. Effect of the value of k on the stress-strain curve.

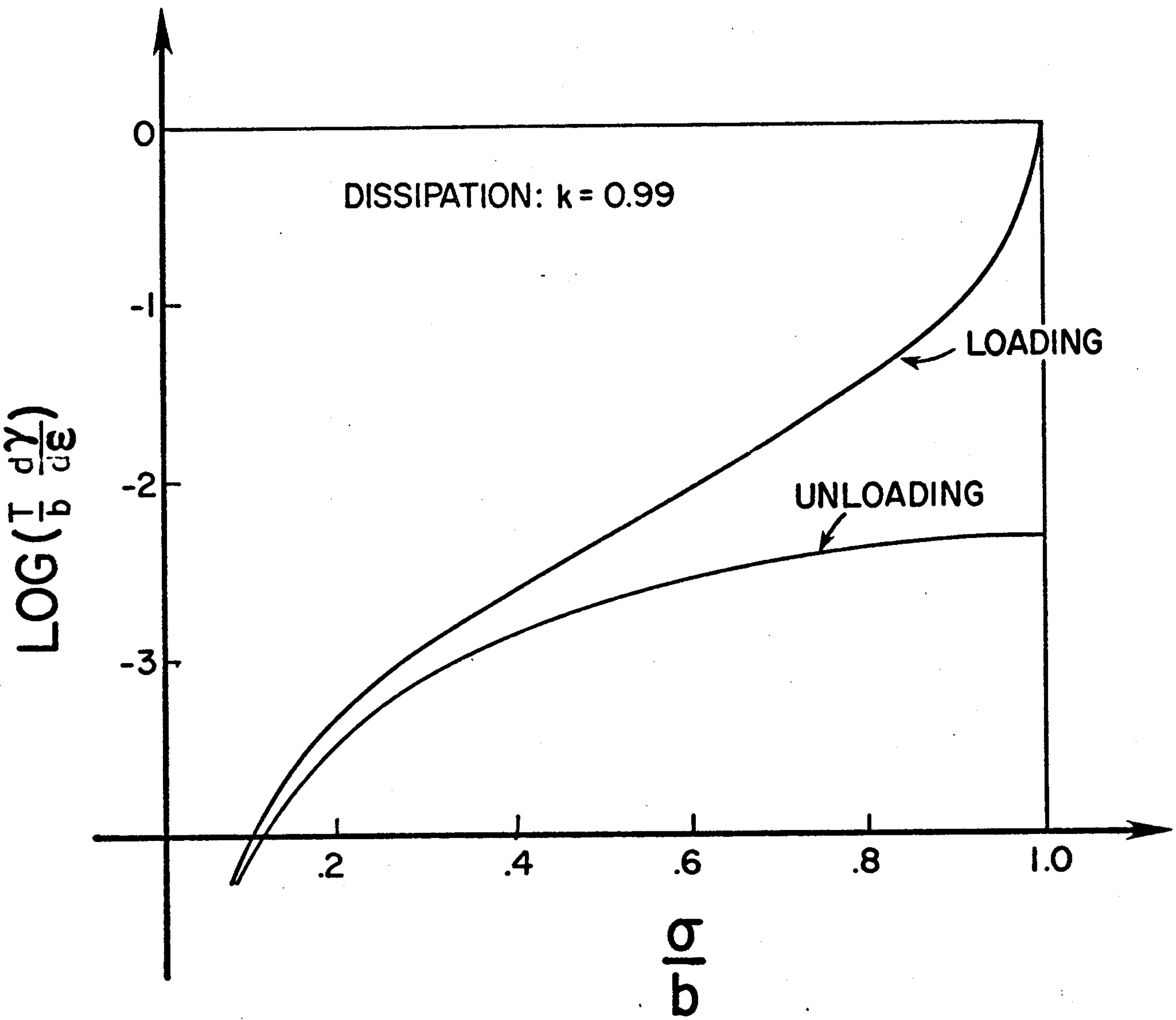

Fig. 5. Dissipation in loading and unloading.

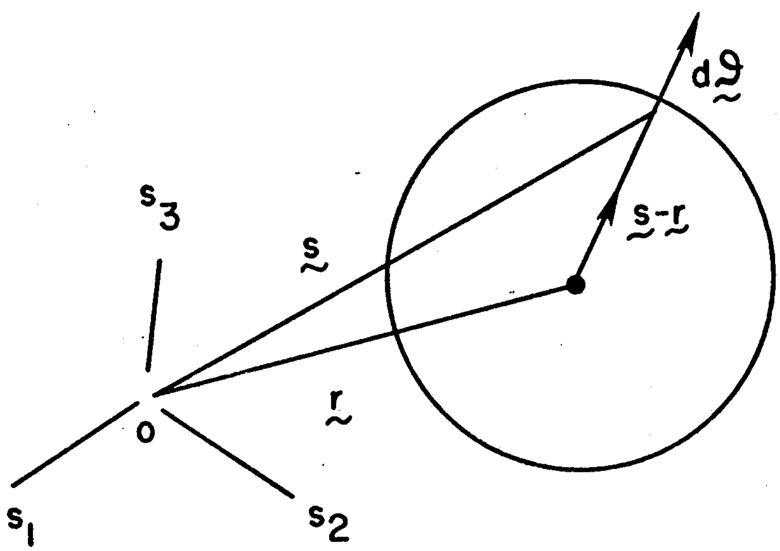

Fig. 6. Geometric illustration of eq.'s (6.41) and (6.42).

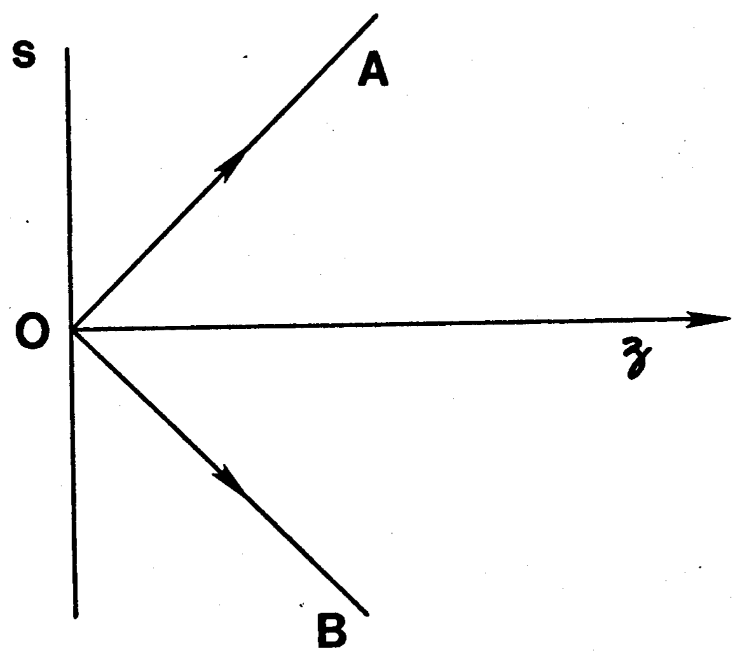

Fig. 7. s plotted vs. ζ showing cusp at $s = \zeta$.

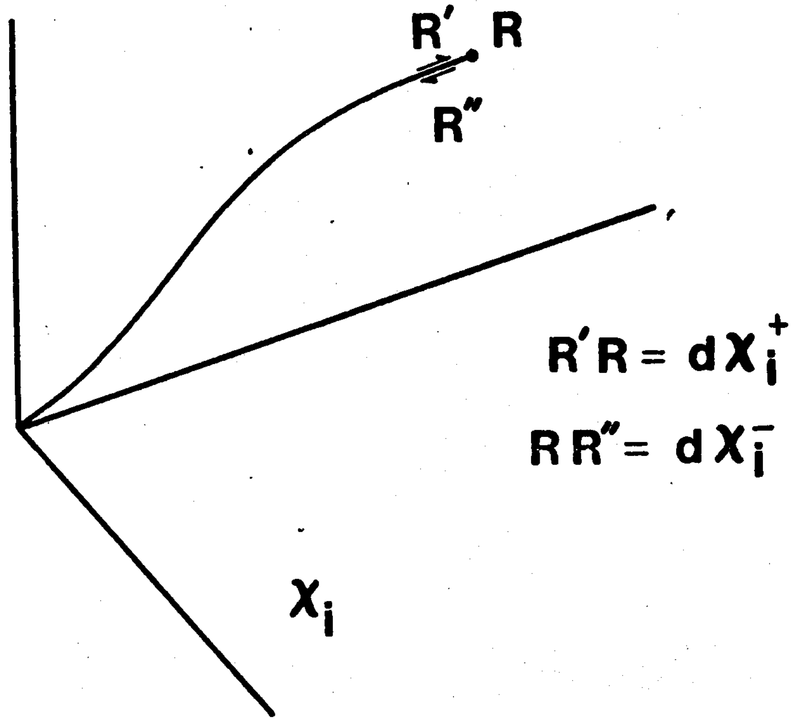

Fig. 8. R as a reversal point.

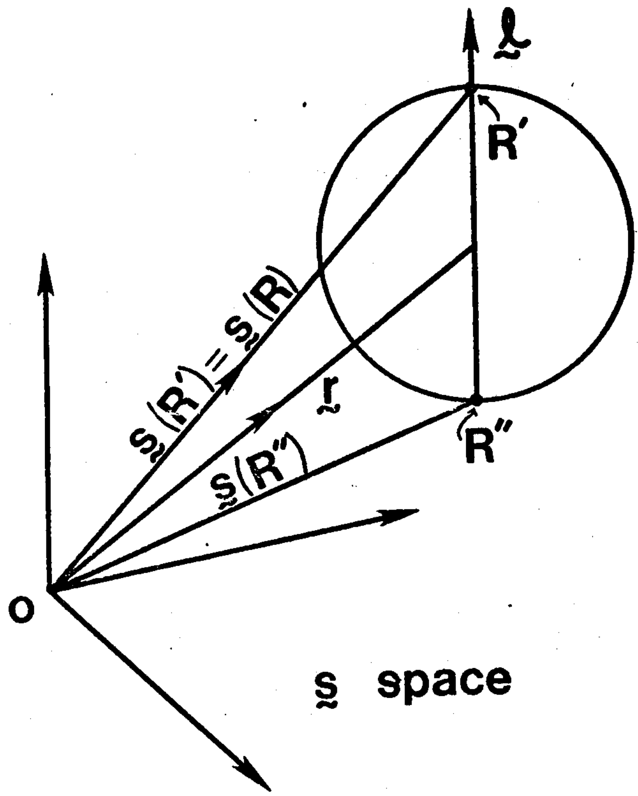

Fig. 9. Stress responses in the limits $R' \to R$, $R'' \to R$.

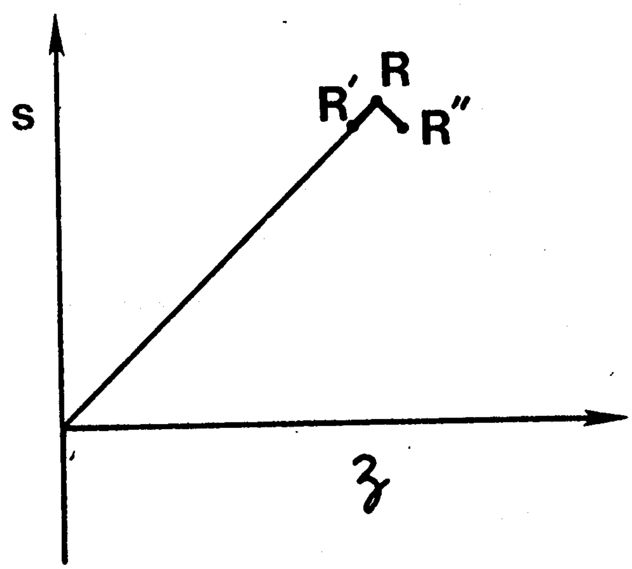

Fig. 10. A history with a point of strain reversal.

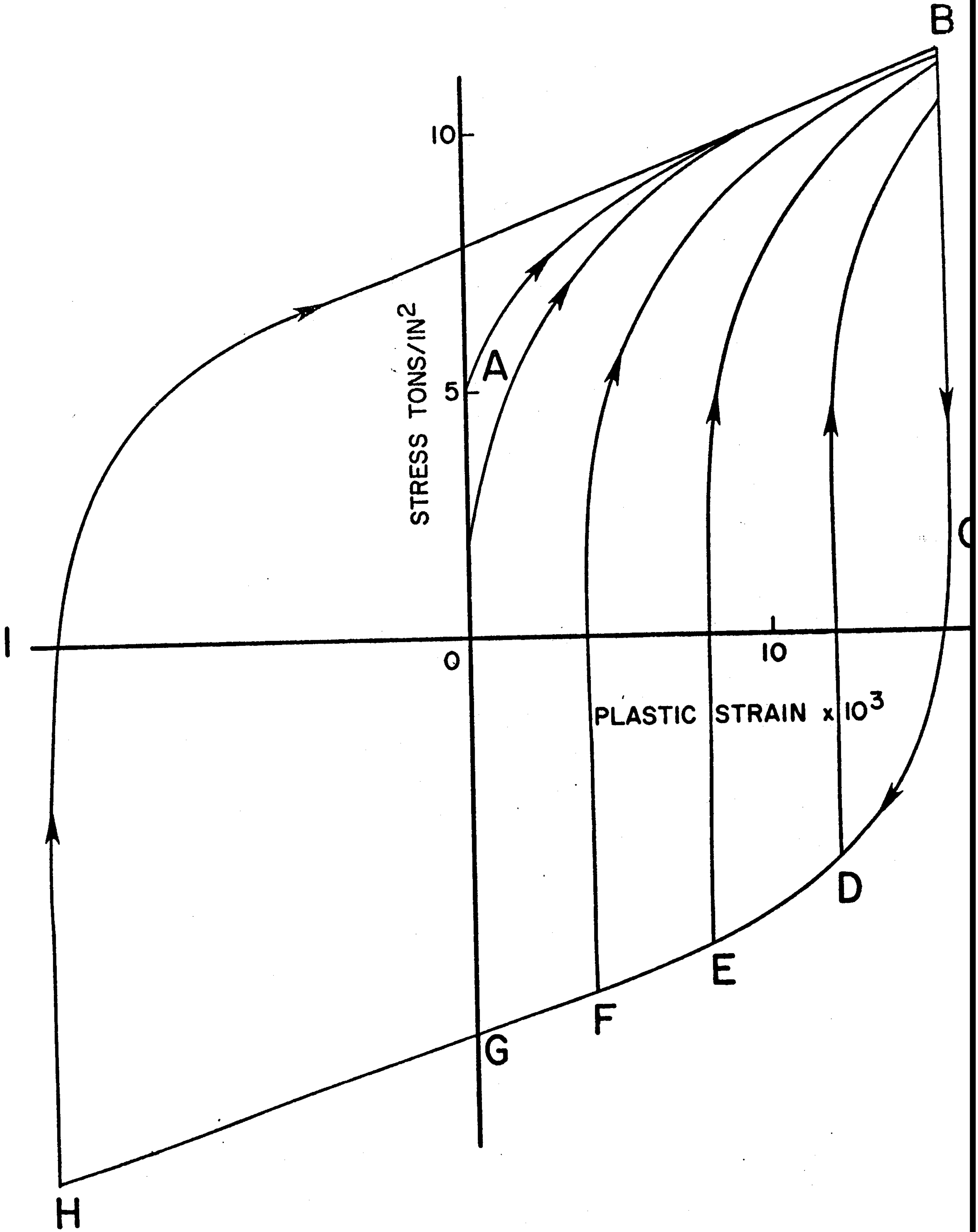

Fig. 11. Theoretically predicted unloading-reloading behavior of aluminum.